Global solvability and Lagrange stability of semilinear differential-algebraic equations and applications to nonlinear radio engineering

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The theory of DAEs

Differential-Algebraic Equations (DAEs) are also called descriptor, algebraic-differential and degenerate differential equations.

Fields of application of the theory of semilinear DAEs are radio engineering, economics, control theory, robotics technology, mechanics, chemical kinetics.

The development of DAE theory: K. Weierstrass (1867), L. Kronecker (1890), V.P. Skripnik (1964), A.G. Rutkas (1975), R.E. Showalter (1975), S.L. Campbell (1976), Yu.E. Boyarintsev (1977), A. Favini (1977), V.F. Chistyakov (1980), L.R. Petzold (1982), L.A. Vlasenko (1987), E. Hairer (1988), P. Kunkel (1991), V. Mehrmann (1991), V.P. Yakovecz (1991), R. März (1994), C. Tischendorf (1994), A.A. Shcheglova (1995), A.M. Samoilenko (2000), R. Riaza (2000).

The research of the DAE global solvability: R. März (1994); C. Tischendorf (1994); L.A. Vlasenko, A.G. Rutkas (2003), A.D. Myshkis (2008); Yu.E. Gliklikh (2014).

The stability of DAEs: L. Dai (1989), R. März (1994), C. Tischendorf (1994), A.A. Shcheglova (2004), V.F. Chistyakov (2004), Yu.E. Boyarintsev (2006), S.L. Campbell (2009), V.H. Linh (2009), Sh. Xu, J. Lam (2006).

Consider **the Cauchy problem for the systems of differential-algebraic equations** which in a vector form have the representation as the **semilinear differential-algebraic equation (DAE)**

$$\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x), \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

- $t,\, t_0 \geq 0, \quad x, x_0 \in \mathbb{R}^n, \quad f(t,x)\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^m$ is a **nonlinear function**;
- $A, B\colon \mathbb{R}^n \to \mathbb{R}^m$ are linear operators (A, B are corresponding $m\times n$ matrices), for $n = m$ the operator A is **degenerate (noninvertible)**;

- denote by $\mathrm{rk}(\lambda A + B)$ the **rank of the pencil** $\lambda A + B$ ($\lambda \in \mathbb{C}$);
- **The pencil** $\lambda A + B$ is called **regular** if $n = m = \mathrm{rk}(\lambda A + B)$ ($n{=}m$, $\det(\lambda A{+}B)\not\equiv 0$). Otherwise, if $n \neq m$ or $n = m$ and $\mathrm{rk}(\lambda A + B) < n$ ($\det(\lambda A + B) \equiv 0$), the pencil is called **singular**.

  Notice that the system of equations corresponding the DAE (1) with the singular pencil may be underdetermined or overdetermined.
- A function $x(t)$ is called a **solution** of the Cauchy problem (1), (2) on some interval $[t_0,t_1)$, $t_1 \leq \infty$, if $x(t) \in C([t_0,t_1),\mathbb{R}^n)$, $Ax(t) \in C^1([t_0,t_1),\mathbb{R}^n)$, $x(t)$ satisfies the equation (1) on $[t_0,t_1)$ and the initial condition (2).

Definitions. The objective of the work

The Cauchy problem (1), (2): $\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x), \quad x(t_0) = x_0$

- A solution $x(t)$ of the Cauchy problem (1), (2) is called **global** if it exists on the whole interval $[t_0,\infty)$.
- A solution $x(t)$ of the Cauchy problem (1), (2) has a **finite escape time** if it exists on some finite interval $[t_0,T)$ and is unbounded, i.e., there exists $T < \infty$ $(T > t_0)$ such that $\lim\limits_{t\to T-0} \|x(t)\| = +\infty$.
- A solution $x(t)$ of the Cauchy problem (1), (2) is called **Lagrange stable** if it is global and bounded, i.e., the solution $x(t)$ exists on $[t_0,\infty)$ and $\sup\limits_{t\in[t_0,\infty)} \|x(t)\| < +\infty$.
- A solution $x(t)$ of the Cauchy problem (1), (2) is called **Lagrange instable** if it has a finite escape time.

The objective of the work is to find conditions of the existence and uniqueness of a global solution, as well as conditions of the boundedness of global solutions, and to find conditions of the existence and uniqueness of a solution with finite escape time for the semilinear DAE (1). The obtained results are applied to a study of the dynamics of nonlinear electrical circuits.

The nonlinear function on the right side of the DAE generates one of the main challenge for the research of the global solvability and stability. To solve this challenge we use, in particular, the **solution extension method**. *The following lemma is the generalization* of La Salle's theorem on extension of solutions of an **ordinary differential equation** (ODE) [J. La Salle, S. Lefschetz, Stability by Liapunov's direct method with applications].

Lemma. Let $f(t,x) \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial}{\partial x} f(t,x)$ be continuous on $[0,\infty) \times \mathbb{R}^n$ and there exist $K(t,v) \in C([0,\infty) \times (0,\infty), \mathbb{R})$ and a positive definite function $V(t,x) \in C^1([0,\infty) \times \mathbb{R}^n, \mathbb{R})$ such that:

1) $V(t,x) \to +\infty$ as $\|x\| \to +\infty$ uniformly in t on every finite interval $[a,b) \subset [0,\infty)$;

2) for any $T > 0$ there exists a bounded set $\Omega_T \in \mathbb{R}^n$, containing the origin, such that $\dot{V}\big|_{(3)} \le K(t, V(t,x))$ $\forall x \in \Omega_T^c$, $t \ge 0$, where $\dot{V}\big|_{(3)} = \frac{\partial V}{\partial t} + (\operatorname{grad} V, f_T(t,x))$

$$\dot{x} = f_T(t,x), \quad f_T(t,x) = \begin{cases} f(t,x), & 0 \le t \le T, \\ f(T,x), & t > T \ (T > 0) \end{cases} \qquad (3)$$

($f_T(t,x)$ is the **truncation** of $f(t,x)$ over t).

3) the differential inequality $\dot{v} \le K(t,v)$, $t \ge 0$, has no positive solution $v(t)$ with finite escape time.

Then every solution $x(t)$ of the ODE $\dot{x} = f(t,x)$ exists on the semiaxis $[t_0,\infty)$.

- A system of one-dimensional projectors $\{\Theta_k\}_{k=1}^{s}$ such that $\Theta_k : Z \to Z$, $\Theta_i \Theta_j = \Theta_j \Theta_i = \delta_{ij} \Theta_i$ (δ_{ij} is the Kronecker delta), and $E_Z = \sum_{k=1}^{s} \Theta_k$ is called an **additive resolution of the identity** in s-dimensional normed linear space Z.
- Let X, Z be s-dimensional normed linear spaces, $D \subset X$. An operator function $\Phi(x) : D \to L(X,Z)$ is called **basis invertible** on the convex hull $\mathrm{conv}\{u,v\}$ of vectors $u, v \in D$ if for any set of vectors $\{x^k\}_{k=1}^{s} \subset \mathrm{conv}\{u,v\}$ and some additive resolution of the identity $\{\Theta_k\}_{k=1}^{s}$ in the space Z the operator $\Lambda = \sum_{k=1}^{s} \Theta_k \Phi(x^k) \in L(X,Z)$ has the inverse operator $\Lambda^{-1} \in L(Z,X)$.

Let us represent the mapping $\Phi(x) : D \to L(X,Z)$ as the matrix relative to some bases in the s-dimensional spaces X, Z:

$$\Phi(x) = \begin{pmatrix} \Phi_{11}(x) & \cdots & \Phi_{1s}(x) \\ \cdots & \cdots & \cdots \\ \Phi_{s1}(x) & \cdots & \Phi_{ss}(x) \end{pmatrix},$$

then the operator Λ has the form

$$\Lambda = \begin{pmatrix} \Phi_{11}(x^1) & \cdots & \Phi_{1s}(x^1) \\ \cdots & \cdots & \cdots \\ \Phi_{s1}(x^s) & \cdots & \Phi_{ss}(x^s) \end{pmatrix}.$$

The case of the regular pencil $\lambda A + B$

Recall that $\lambda A + B$ is regular if $n = m = \mathrm{rk}(\lambda A + B)$.

Let $\lambda A + B$ be a regular **pencil of index 1**, that is

$$\exists C_1, C_2 > 0 : \left\|(\lambda A + B)^{-1}\right\| \le C_1 \quad \forall \lambda \in \mathbb{C} : |\lambda| \ge C_2. \tag{4}$$

Then there exist **the spectral projectors of Riss type** [A.G. Rutkas, Cauchy problem for the equation $Ax'(t) + Bx(t) = f(t)$]:

$$\begin{aligned} P_1 &= \frac{1}{2\pi i} \oint_{|\lambda|=C_2} (\lambda A + B)^{-1} A\, d\lambda,\ P_2 = E - P_1 \\ Q_1 &= \frac{1}{2\pi i} \oint_{|\lambda|=C_2} A\, (\lambda A + B)^{-1} d\lambda,\ Q_2 = E - Q_1, \end{aligned} \quad (E \text{ is an identity operator}) \tag{5}$$

which decompose the space $\mathbb{R}^n$ into direct sums of subspaces:

$$\mathbb{R}^n = X_1 \dot{+} X_2 = Y_1 \dot{+} Y_2, \quad X_j = P_j \mathbb{R}^n, Y_j = Q_j \mathbb{R}^n, \quad j = 1, 2. \tag{6}$$

$A_j = A|_{X_j} : X_j \to Y_j$, $B_j = B|_{X_j} : X_j \to Y_j$, $j = 1,2$ are such that $A_2 = 0$, $\exists A_1^{-1} \in L(Y_1, X_1)$ (if $X_1 \ne \{0\}$), $\exists B_2^{-1} \in L(Y_2, X_2)$ (if $X_2 \ne \{0\}$).

The invertible **auxiliary operator** $G = AP_1 + BP_2 = Q_1 A + Q_2 B$ [L.A. Vlasenko, Implicit linear time-dependent differential-difference equations and applications].

The method of Riss type spectral projectors

Applying Q_1, Q_2 and G^{-1} to the DAE (1) $\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x)$, we obtain the equivalent system

$$\begin{cases} \dfrac{d}{dt}(P_1 x) + G^{-1}BP_1 x = G^{-1}Q_1 f(t,x), \\ G^{-1}Q_2 f(t,x) - P_2 x = 0. \end{cases}$$

The Cauchy problem for the DAE (1), (2): $\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x)$, $x(t_0) = x_0$.

Theorem 1. Let $f(t,x) \in C([0,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\frac{\partial}{\partial x}f(t,x)$ be continuous on $[0,\infty)\times\mathbb{R}^n$, $\lambda A + B$ be a **regular pencil** of index 1 and

1) $\forall t \geq 0 \forall x_1 \in X_1 \exists x_2 \in X_2 : (t, x_1 + x_2) \in L_0 = \{(t,x) \in [0,\infty)\times\mathbb{R}^n \mid Q_2[Bx - f(t,x)] = 0\}$;

2) $\forall (t, x_1 + u_i) \in L_0$, $i = 1,2$, where $x_1 \in X_1$, $u_i \in X_2$, the operator function $\Phi(u) = \left[\frac{\partial}{\partial x}(Q_2 f(t, x_1 + u)) - B\right] P_2 : X_2 \to L(X_2, Y_2)$ be basis invertible on the convex hull $\mathrm{conv}\{u_1, u_2\}$;

3) there exist a self-adjoint positive operator $H \in L(X_1)$ and for each $T > 0$ there exist a number $R_T > 0$ such that

$$(HP_1 x, G^{-1}Q_1 f(t,x)) \leq 0 \quad \forall (t,x) \in L_0 : 0 \leq t \leq T,\ \|P_1 x\| \geq R_T. \qquad (7)$$

Then **for every initial point** $(t_0, x_0) \in L_0$ **there exists a unique solution** $x(t)$ **of the Cauchy problem** (1), (2) **on** $[t_0, \infty)$.

Corollary 1. Suppose that in Theorem 1 the projection $Q_1 f$ is of the form $Q_1 f(t,x) = S_1(t) P_1 x + \psi(t,x) + e(t)$, where $S_1(t) \in C([0,\infty), L(X_1, Y_1))$, $\psi(t,x) \in C([0,\infty)\times\mathbb{R}^n, Y_1)$, $\frac{\partial \psi(t,x)}{\partial x}$ is continuous on $[0,\infty)\times\mathbb{R}^n$, $e(t) \in C([0,\infty),\mathbb{R}^n)$.

Then Theorem 1 remains valid if instead of (7) the condition $(HP_1 x, G^{-1}\psi(t,x)) \leq 0\ \forall (t,x) \in L_0 : 0 \leq t \leq T,\ \|P_1 x\| \geq R_T$ is fulfilled.

Notice that if $n = m$ and the operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is invertible (nondegenerate), then the semilinear DAE (1) $\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x)$ is equivalent to the ODE $\frac{d}{dt}x(t) + A^{-1}Bx(t) = A^{-1}f(t,x)$.

In this case Corollary 1 may be represented in the following form.

Theorem (*The existence and uniqueness theorem of the ODE global solution*). Let $F(t,x) \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial F(t,x)}{\partial x}$ be continuous on $[0,\infty) \times \mathbb{R}^n$, S is a real $n \times n$ matrix, $e(t) \in C([0,\infty),\mathbb{R}^n)$ and
there exist the matrix $H = H^* > 0$ and for each $T > 0$ there exist $R_T > 0$ such that

$$(Hx,F(t,x)) \leq 0, \qquad 0 \leq t \leq T, \quad \|x\| \geq R_T.$$

Then for every initial point $(t_0,x_0) \in [0,\infty) \times \mathbb{R}^n$ there exists a unique solution of the **Cauchy problem for the ODE** $\frac{d}{dt}x(t) = Sx(t) + F(t,x) + e(t)$, $x(t_0) = x_0$ on $[t_0,\infty)$.

The model of a hybrid four-pole radio engineering filter

An input current I_1, an input voltage U_1, nonlinear resistances φ_1, φ_2, φ_3, a nonlinear conductance h, linear resistances r_1, r_2, r_3, a linear conductance g, an inductance L and a capacitance C are given.

Let $I_1(t), U_1(t) \in C([0,\infty),\mathbb{R})$, $h(y), \varphi_1(y), \varphi_2(y), \varphi_3(y) \in C^1(\mathbb{R},\mathbb{R})$, r_1, r_2, r_3, g, L, $C > 0$.

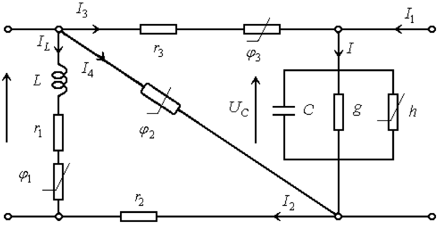

Fig. 1. The electric circuit diagram

The model of the circuit Fig. 1 is described by the system with the variables $x_1 = I_L$, $x_2 = U_C$, $x_3 = I_3$, $x_4 = I_4$:

$$L\frac{d}{dt}x_1 + r_1 x_1 = U_1(t) - \varphi_1(x_1), \qquad (8)$$

$$C\frac{d}{dt}x_2 + g x_2 - x_3 = I_1(t) - h(x_2), \qquad (9)$$

$$-x_2 - r_3 x_3 = \varphi_3(x_3) - \varphi_2(x_4), \qquad (10)$$

$$r_2(x_3 + x_4) = U_1(t) - \varphi_2(x_4). \qquad (11)$$

The vector form of the system is the DAE

$$\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x), \qquad (12)$$

where $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$

$$A = \begin{pmatrix} L & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & g & -1 & 0 \\ 0 & -1 & -r_3 & 0 \\ 0 & 0 & r_2 & r_2 \end{pmatrix}$$

$$f(t,x) = \begin{pmatrix} U_1(t) - \varphi_1(x_1) \\ I_1(t) - h(x_2) \\ \varphi_3(x_3) - \varphi_2(x_4) \\ U_1(t) - \varphi_2(x_4) \end{pmatrix}$$

It is easy to verify that $\lambda A + B$ is a regular pencil of index 1.

The projection matrices P_k, Q_k, $k = 1,2$, and the matrix G^{-1} have the form

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -r_3^{-1} & 0 & 0 \\ 0 & r_3^{-1} & 0 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & r_3^{-1} & 1 & 0 \\ 0 & -r_3^{-1} & 0 & 1 \end{pmatrix}, Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -r_3^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r_3^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, G^{-1} = \begin{pmatrix} L^{-1} & 0 & 0 & 0 \\ 0 & C^{-1} & -(r_3 C)^{-1} & 0 \\ 0 & -(r_3 C)^{-1} & -r_3^{-1} + r_3^{-2} C^{-1} & 0 \\ 0 & (r_3 C)^{-1} & r_3^{-1} - r_3^{-2} C^{-1} & r_2^{-1} \end{pmatrix}.$$

The projections of the vector x have the form

$$z = P_1 x = \begin{pmatrix} x_1 \\ x_2 \\ -r_3^{-1} x_2 \\ r_3^{-1} x_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \quad u = P_2 x = \begin{pmatrix} 0 \\ 0 \\ x_3 + r_3^{-1} x_2 \\ x_4 - r_3^{-1} x_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

Conclusions. *Let the following conditions be fulfilled:*

1) $\forall t \geq 0 \forall z_2 \in \mathbb{R} \exists u_3, u_4 \in \mathbb{R}$ such that

$$\begin{cases} -r_3 u_3 = \varphi_3(u_3 - r_3^{-1} z_2) - \varphi_2(u_4 + r_3^{-1} z_2), \\ r_2(u_3 + u_4) = U_1(t) - \varphi_2(u_4 + r_3^{-1} z_2); \end{cases} \tag{13}$$

2) for any $v, w \in X_2$ satisfying (13) under $u = v$ and $u = w$, the condition $[\varphi_2'(u_4^2 + \xi) + r_2]\,[\varphi_3'(u_3^1 - \xi) + r_3] + r_2 \varphi_2'(u_4^1 + \xi) \neq 0$, $\xi \in \mathbb{R}$, is fulfilled for any $u^k = (u_1^k, u_2^k, u_3^k, u_4^k)^T \in \operatorname{conv}\{v, w\}$, $k = 1, 2$;

3) $-x_1 \varphi_1(x_1) - 3x_2 h(x_2) + 3r_3^{-1} x_2^2 + 3x_2 x_3 \leq 0$ $\forall x \in \mathbb{R}^4 \colon \sqrt{x_1^2 + x_2^2(1 + \frac{2}{r_3^2})} \geq R_T$, (11).

Then by Corollary 1 for every initial point $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^4$ satisfying the algebraic equations (10), (11) there exists a unique solution of the Cauchy problem for the DAE (12), $x(t_0) = x_0$ on $[t_0, \infty)$ $(x(t_0) = (I_L(t_0), U_C(t_0), I_3(t_0), I_4(t_0))^T)$.

It means that for any initial moment t_0 and any initial currents and voltages $I_L(t_0)$, $U_C(t_0)$, $I_3(t_0)$, $I_4(t_0)$ *satisfying* $-U_C(t_0) - r_3 I_3(t_0) = \varphi_3(I_3(t_0)) - \varphi_2(I_4(t_0))$, $r_2(I_3(t_0) + I_4(t_0)) = U_1(t_0) - \varphi_2(I_4(t_0))$ *there exists a unique distribution of the currents and voltages in the circuit Fig. 1 for all* $t \geq t_0$.

The particular case. The conditions of Corollary 1 are satisfied for $\varphi_1(y) = \alpha_1 y^{2m+1}$, $h(y) = \alpha_2 y^{2s+1}$, $\varphi_3(y) = \alpha_3 y^{2n-1}$, $\varphi_2(y) = \alpha_4 y^{2r-1}$, $y \in \mathbb{R}$, $m, n, r, s \in \mathbb{N}$, $\alpha_k > 0, k = \overline{1,4}$.

The Cauchy problem (1)**,** (2): $\dfrac{d}{dt}[Ax(t)] + Bx(t) = f(t,x), \quad x(t_0) = x_0.$

- A solution $x(t)$ of the Cauchy problem (1), (2) is called **Lagrange stable** if it is global and bounded, i.e., the solution $x(t)$ exists on $[t_0,\infty)$ and $\sup\limits_{t\in[t_0,\infty)} \|x(t)\| < +\infty$.
- **The equation** (1) **is Lagrange stable** if every solution of the Cauchy problem (1), (2) is Lagrange stable.
- A solution $x(t)$ of the Cauchy problem (1), (2) has a **finite escape time** if it exists on some finite interval $[t_0,T)$ and is unbounded ($\lim\limits_{t\to T-0} \|x(t)\| = +\infty$).
- **The equation** (1) **is Lagrange instable** if every solution of the Cauchy problem (1), (2) has a finite escape time.

The Cauchy problem (1), (2): $\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x)$, $x(t_0) = x_0$.

Theorem 2. Let $f(t,x) \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\frac{\partial}{\partial x} f(t,x)$ be continuous on $[0,\infty) \times \mathbb{R}^n$, $\lambda A + B$ be a **regular pencil** of index 1 and

1) $\forall t \geq 0 \, \forall x_1 \in X_1 \, \exists x_2 \in X_2 : (t, x_1 + x_2) \in L_0 = \{(t,x) \in [0,\infty) \times \mathbb{R}^n \mid Q_2[Bx - f(t,x)] = 0\}$;

2) $\forall (t, x_1 + u_i) \in L_0$, $i = 1,2$, where $x_1 \in X_1$, $u_i \in X_2$, the operator function $\Phi(u) = \left[\frac{\partial}{\partial x}(Q_2 f(t, x_1 + u)) - B\right] P_2 : X_2 \to L(X_2, Y_2)$ be basis invertible on $\mathrm{conv}\{u_1, u_2\}$;

3) for some self-adjoint positive operator $H \in L(X_1)$ and some number $R > 0$ there exist functions $k(t) \in C([0,\infty), \mathbb{R})$, $U(v) \in C((0,\infty), (0,\infty))$ such that $v = \frac{1}{2}(HP_1x, P_1x)$, $\int\limits_c^{+\infty} \frac{dv}{U(v)} = +\infty$ $(c > 0)$ and

$$(HP_1x, G^{-1}[-BP_1x + Q_1 f(t,x)]) \leq k(t) U(v) \quad \forall (t,x) \in L_0 : \|P_1x\| \geq R.$$

Then **for every initial point** $(t_0, x_0) \in L_0$ **there exists a unique solution** $x(t)$ **of the Cauchy problem** (1), (2) **on** $[t_0, \infty)$.

If, additionally, $\int\limits_{t_0}^{+\infty} k(t)\,dt < +\infty$ and $\exists C, M > 0$ such that $\|G^{-1} Q_2 f(t, P_1x)\| \leq C$ $\forall t \in [0,\infty), \|P_1x\| \leq M$, then for the initial points $(t_0, x_0) \in L_0$ **the equation** (1) **is Lagrange stable**.

The Cauchy problem (1), (2): $\frac{d}{dt}[Ax(t)]+Bx(t)=f(t,x)$, $x(t_0)=x_0$.

Theorem 3. Let $f(t,x)\in C([0,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\frac{\partial}{\partial x}f(t,x)$ be continuous on $[0,\infty)\times\mathbb{R}^n$, $\lambda A+B$ be a **regular pencil** of index 1 and

1) $\forall t\geq 0\,\forall x_1\in X_1\,\exists x_2\in X_2:(t,x_1+x_2)\in L_0=\{(t,x)\in[0,\infty)\times\mathbb{R}^n\,|\,Q_2[Bx-f(t,x)]=0\}$;

2) $\forall(t,x_1+u_i)\in L_0$, $i=1,2$, where $x_1\in X_1$, $u_i\in X_2$, the operator function $\Phi(u)=\left[\frac{\partial}{\partial x}(Q_2f(t,x_1+u))-B\right]P_2:X_2\to L(X_2,Y_2)$ be basis invertible on $\mathrm{conv}\{u_1,u_2\}$;

3) there exist a region $\Omega\subset X_1$ such that $P_1x=0\notin\Omega$ and the component $P_1x(t)$ of every existing solution $x(t)$ with the initial point $(t_0,x_0)\in L_0$, where $P_1x_0\in\Omega$, remains all the time in Ω;

4) for some self-adjoint positive operator $H\in L(X_1)$ there exist functions $k(t)\in C([0,\infty),\mathbb{R})$, $U(v)\in C((0,\infty),(0,\infty))$ such that $v=\frac{1}{2}(HP_1x,P_1x)$, $\int_c^{+\infty}\frac{dv}{U(v)}<+\infty$ $(c>0)$, $\int_{t_0}^{+\infty}k(t)\,dt=\infty$ and

$$(HP_1x,G^{-1}[-BP_1x+Q_1f(t,x)])\geq k(t)U(v)\quad\forall(t,x)\in L_0:P_1x\in\Omega.$$

Then **for every initial point** $(t_0,x_0)\in L_0$, where $P_1x_0\in\Omega$, **there exists a unique solution of the Cauchy problem** (1), (2) and **this solution has a finite escape time**.

Consider the *Cauchy problem* (1), (2): $\frac{d}{dt}[Ax(t)]+Bx(t)=f(t,x),\ \ x(t_0)=x_0$, where $\lambda A+B$ is a *regular pencil of index 1*.

$\{t_i=t_0+ih, i=0,\dots,N, t_N=T\}$ is a uniform grid on an interval $[t_0,T]$ $(h=(T-t_0)/N)$.

$z=P_1x$, $u=P_2x$ are the projections of the vector $x=z+u\in\mathbb{R}^n$ onto the subspaces X_1, X_2 correspondingly;

$x_i=z_i+u_i$, $i=0,\dots,N$ are the values of an approximate solution of the problem at t_i.

E is the identity $n\times n$ matrix.

Initial values z_0, u_0 are chosen so that *the consistency condition* $u_0=G^{-1}Q_2f(t_0,z_0+u_0)$ is satisfied (the condition $(t_0,x_0)\in L_0$ of the theorems).

The numerical method:

$$x_0=z_0+u_0,\quad z_{i+1}=(E-hG^{-1}B)z_i+hG^{-1}Q_1f(t_i,z_i+u_i), \tag{14}$$

$$\begin{aligned} u_{i+1}=\Big[E-G^{-1}Q_2\tfrac{\partial}{\partial x}f(t_{i+1},z_{i+1}+u_i)\Big]^{-1}G^{-1}Q_2\times \\ \times\Big[f(t_{i+1},z_{i+1}+u_i)-\tfrac{\partial}{\partial x}f(t_{i+1},z_{i+1}+u_i)\,u_i\Big], \end{aligned} \tag{15}$$

$$x_{i+1}=z_{i+1}+u_{i+1},\quad i=0,\dots,N-1, \tag{16}$$

The spectral projectors of Riss type can be calculated by the formulas

$$P_1=\operatorname*{Res}_{\mu=0}\left(\frac{(A+\mu B)^{-1}A}{\mu}\right),\ Q_1=\operatorname*{Res}_{\mu=0}\left(\frac{A(A+\mu B)^{-1}}{\mu}\right),\ P_2=E-P_1,\ Q_2=E-Q_1$$

Input currents I_1, I_2,
nonlinear resistances φ_1, φ_2,
nonlinear conductances h_1, h_2,
linear resistances r_1, r_2,
a linear conductance g,
an inductance L and
a capacitance C are given.

Let $I_1(t), I_2(t) \in C([0,\infty),\mathbb{R})$, $\varphi_1(y), \varphi_2(y), h_1(y), h_2(y) \in C^1(\mathbb{R},\mathbb{R})$, r_1, r_2, g, L, $C > 0$.

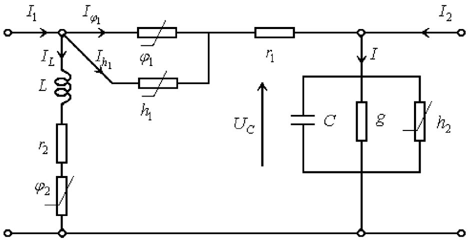

Fig. 2. The electric circuit diagram

The model of Fig. 2 is described by the system with the variables $x_1 = I_{\varphi_1}$, $x_2 = I_L$, $x_3 = U_C$, $\gamma(x_1) = h_1(\varphi_1(x_1))$:

$$L\frac{d}{dt}x_2 - r_1 x_1 + r_2 x_2 - x_3 = \varphi_1(x_1) + r_1\gamma(x_1) - \varphi_2(x_2), \quad (17)$$

$$C\frac{d}{dt}x_3 - x_1 + g x_3 = I_2(t) + \gamma(x_1) - h_2(x_3), \quad (18)$$

$$x_1 + x_2 = I_1(t) - \gamma(x_1). \quad (19)$$

$$A = \begin{pmatrix} 0 & L & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -r_1 & r_2 & -1 \\ -1 & 0 & g \\ 1 & 1 & 0 \end{pmatrix}$$

The vector form of the system is the DAE

$$\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x), \quad (20)$$

where $x = (x_1,x_2,x_3)^T \in \mathbb{R}^3$

$$f(t,x) = \begin{pmatrix} \varphi_1(x_1) + r_1\gamma(x_1) - \varphi_2(x_2) \\ I_2(t) + \gamma(x_1) - h_2(x_3) \\ I_1(t) - \gamma(x_1) \end{pmatrix}$$

It is easy to verify that $\lambda A+B$ is a regular pencil of index 1.

Let the following conditions be fulfilled:

1) $\forall t\geq 0 \forall x_2\in\mathbb{R}\exists w\in\mathbb{R}$ such that

$$w=I_1(t)-\gamma(w-x_2). \tag{21}$$

2) for any $w_1, w_2\in\mathbb{R}$ satisfying (21) the condition $\gamma'(w-x_2)\neq -1$ is fulfilled for any $w\in \operatorname{conv}\{w_1,w_2\}$, $x_2\in\mathbb{R}$;

3) there exist $R>0$ and functions $k(t)\in C([0,\infty),\mathbb{R})$, $U(v)\in C((0,\infty),(0,\infty))$ such that $v=Lx_2^2+Cx_3^2$, $\int\limits_{c}^{+\infty}\frac{dv}{U(v)}=\infty$ and $-(r_1+r_2)x_2^2-gx_3^2-x_2\varphi_2(x_2)-$ $-x_3h_2(x_3)+x_2\varphi_1(x_1)+r_1x_2I_1(t)+x_3I_2(t)\leq k(t)U(v)\ \ \forall t\geq 0, \sqrt{2x_2^2+x_3^2}\geq R.$

Then by Theorem 2 for every initial point $(t_0,x^0)\in[0,\infty)\times\mathbb{R}^3$ satisfying $x_1^0+x_2^0=I_1(t)-\gamma(x_1^0)$ there exists a unique solution $x(t)$ of the Cauchy problem for the DAE (20), $x(t_0)=x^0$ on $[t_0,\infty)$
$(x^0=(x_1^0,x_2^0,x_3^0)^T,\ \ x(t_0)=(I_{\varphi_1}(t_0),I_L(t_0),U_C(t_0))^T)$.

If, additionally, $\int\limits_{t_0}^{+\infty}k(t)\,dt<+\infty$ and $\exists C,M>0:\sup\limits_{t\in[0,\infty)}\max\limits_{|z_1|\leq M}|I_1(t)-\gamma(z_1)|\leq C$, then for the initial points (t_0,x^0) satisfying $x_1^0+x_2^0=I_1(t)-\gamma(x_1^0)$ *the equation* (20) *is Lagrange stable.*

It means that if the aforementioned conditions are fulfilled, then for any initial moment t_0 and any initial currents and voltage $I_{\varphi_1}(t_0)$, $I_L(t_0)$, $U_C(t_0)$ satisfying $I_{\varphi_1}(t_0) + I_L(t_0) = I_1(t_0) - \gamma(I_{\varphi_1}(t_0))$ there exists a unique distribution of the currents and voltages in the circuit Fig. 2 for all $t \geq t_0$.

The currents and voltages for the obtained distribution are bounded for all $t \geq t_0$ (Lagrange stability) if $\int_{t_0}^{+\infty} k(t)\,dt < +\infty$ and there exist $M_1, M_2 > 0$ such that $\sup_{t \in [0,\infty)} \max_{|z_1| \leq M_2} |I_1(t) - \gamma(z_1)| \leq M_1$.

The particular cases.

$$\begin{gathered}\varphi_1(y) = \alpha_1 y^3, \varphi_2(y) = \alpha_2 y^3, \\ h_2(y) = \alpha_3 y^3, \gamma(y) = h_1(\varphi_1(y)) = \alpha_4 y^9, \alpha_k > 0;\end{gathered} \tag{22}$$

$$\begin{gathered}\varphi_1(y) = \alpha_1 \sin(y), \varphi_2(y) = \alpha_2 \sin(y), \\ h_2(y) = \alpha_3 \cos(y), \gamma(y) = 0.5\cos(\cos(y)), \alpha_k > 0, y \in \mathbb{R}\end{gathered} \tag{23}$$

The conditions of Theorem 2 (the aforementioned conditions) are fulfilled for the nonlinear resistances and conductances of the form (22) or (23).

The existing currents and voltages are bounded if

$\sup\limits_{t\in[0,\infty)} |I_1(t)| < \infty$ and $\sup\limits_{t\in[0,\infty)} |I_2(t)| < \infty$ or $\int\limits_{t_0}^{+\infty} |I_2(t)|\,dt < +\infty$.

In particular, these requirements are fulfilled for input currents of the form

$$I_k(t) = b_k e^{-a_k t}, I_k(t) = b_k e^{-\frac{(t-d_k)^2}{\sigma_k^2}}, I_k(t) = b_k \sin(\omega_k t + \theta_k), \tag{24}$$

where $a_k > 0$, b_k, σ_k, d_k, $\omega_k \in \mathbb{R}$, $\theta_k \in [0, 2\pi]$, $k = 1{,}2$.

$L = 0.5$ nH, $C = 0.4$ pF, $r_1 = 0.02$ ohm, $r_2 = 0.01$ ohm, $g = 0.2$ ohm^{-1}.
$\varphi_1(y) = \sin(y)$, $\varphi_2(y) = \sin(y)$, $h_2(y) = \cos(y)$, $\gamma(y) = 0.5\cos(\cos(y))$,
$I_1(t) = 50\sin(0.5t - 1.6)$, $I_2(t) = 50\sin(t)$, $t_0 = 0$, $x_0 = (0, -50.2488, 0)^T$.

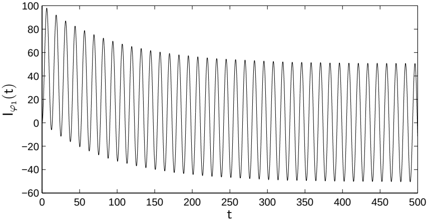

Fig. 3. The current $I_{\varphi_1}(t)$

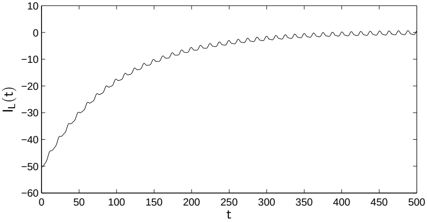

Fig. 4. The current $I_L(t)$

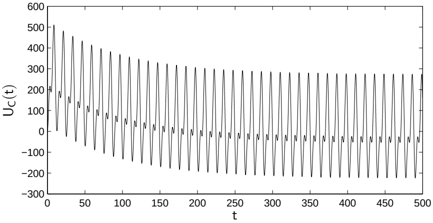

Fig. 5. The current $I_L(t)$

The model of an impedance four-pole radio engineering filter

$L = 0.5$ nH, $C = 0.4$ pF, $r_1 = 0.02$ ohm, $r_2 = 0.01$ ohm, $g = 0.2$ ohm^{-1}.
$\varphi_1(y) = 0.1y^3$, $\varphi_2(y) = 0.1y^3$, $h_2(y) = 0.1y^3$, $\gamma(y) = 0.1y^9$,
$I_1(t) = 0.1t$, $I_2(t) = 0.001t^2$, $t_0 = 0$, $x_0 = (0,0,0)^T$.

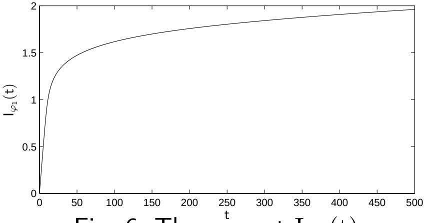

Fig. 6. The current $I_{\varphi_1}(t)$

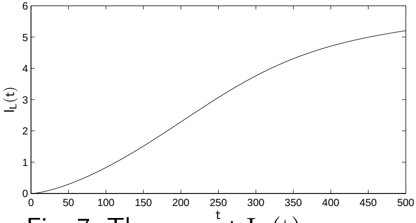

Fig. 7. The current $I_L(t)$

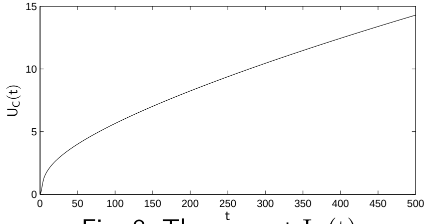

Fig. 8. The current $I_L(t)$

A voltage source $e(t)$,
nonlinear resistances φ, φ_0, ψ,
a nonlinear conductance h,
a linear resistance r,
a linear conductance g,
an inductance L and
a capacitance C are given.

Let $e(t) \in C([0,\infty),\mathbb{R})$, $\varphi(y), \varphi_0(y), \psi(y), h(y) \in C^1(\mathbb{R},\mathbb{R})$, $r,\ g,\ L,\ C > 0$.

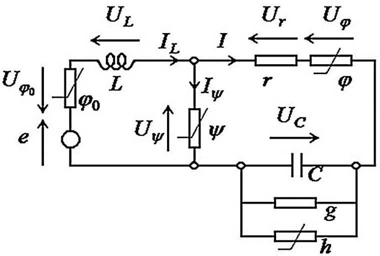

Fig. 9. The electric circuit diagram

The model of the circuit Fig. 9 is described by the system with the variables $x_1 = I_L$, $x_2 = U_C$, $x_3 = I$:

$$L\frac{d}{dt}x_1 + x_2 + r x_3 = e(t) - \varphi_0(x_1) - \varphi(x_3), \quad (25)$$

$$C\frac{d}{dt}x_2 + g x_2 - x_3 = -h(x_2), \quad (26)$$

$$x_2 + r x_3 = \psi(x_1 - x_3) - \varphi(x_3). \quad (27)$$

$$A = \begin{pmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & r \\ 0 & g & -1 \\ 0 & 1 & r \end{pmatrix}$$

The vector form of the system is the DAE

$$\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x), \qquad (28)$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$

$$f(t,x) = \begin{pmatrix} e(t) - \varphi_0(x_1) - \varphi(x_3) \\ -h(x_2) \\ \psi(x_1 - x_3) - \varphi(x_3) \end{pmatrix}$$

It is easy to verify that $\lambda A + B$ is a regular pencil of index 1.

Let the following conditions be fulfilled:

1) for any $a, b \in \mathbb{R}$ there exists $w \in \mathbb{R}$ such that

$$rw = \psi(a - b - w) - \varphi(b + w); \qquad (29)$$

2) for any $a, b, w_1, w_2 \in \mathbb{R}$ satisfying (29) the condition $\psi'(a - b - w) + \varphi'(b + w) \neq -r$ is fulfilled for any $w \in \mathrm{conv}\{w_1, w_2\}$;

3) there exist $R > 0$ and functions $k(t) \in C([0,\infty),\mathbb{R})$, $U(v) \in C((0,\infty),(0,\infty))$ such that $v = Lx_1^2 + Crx_2^2$, $\int\limits_c^{+\infty} \frac{dv}{U(v)} = +\infty$ and $-(gr+1)x_2^2 - x_1\varphi_0(x_1) + $ $+(x_2 - x_1)\psi(x_1 - x_3) - rx_2h(x_2) - x_2\varphi(x_3) + x_1e(t) \leq k(t)U(v)$ for any $(t,x) \in [0,\infty) \times \mathbb{R}^3$ such that $\sqrt{x_1^2 + (1 + r^{-2})x_2^2} \geq R$, (27).

Then by Theorem 2 for any initial moment t_0 and any initial currents and voltage $I_L(t_0)$, $U_C(t_0)$, $I(t_0)$ satisfying $U_C(t_0) + rI(t_0) = \psi(I_L(t_0) - I(t_0)) -$ $-\varphi(I(t_0))$ there exists a unique distribution of the currents and voltages in the circuit Fig. 9 for all $t \geq t_0$.

The currents and voltages for the obtained distribution are bounded for all $t \geq t_0$ (Lagrange stability) if $\int\limits_{t_0}^{+\infty} k(t)\,dt < \infty$.

The particular cases.

$$\varphi_0(y) = \alpha_1 y^{2k-1},\ \varphi(y) = \alpha_2 y^{2l-1},\ \psi(y) = \alpha_3 y^{2j-1},\ h(y) = \alpha_4 y^{2s-1}, \tag{30}$$

$\alpha_i > 0,\ k,r,j,s \in \mathbb{N}$;

$$\varphi_0(y) = \alpha_1 y^{2k-1},\ \varphi(y) = \alpha_2 \sin y,\ \psi(y) = \alpha_3 \sin y,\ h(y) = \alpha_4 \sin y, \tag{31}$$

$\alpha_i > 0,\ k \in \mathbb{N}$.

The conditions of Theorem 2 (the aforementioned conditions) are fulfilled for the nonlinear resistances and conductances of the form (30) if $j \leq k$, $j \leq s$ and α_3 is sufficiently small, and for the nonlinear resistances and conductances of the form (31) if $\alpha_2 + \alpha_3 < r$.

The existing currents and voltages are bounded (in both cases) if

$\sup\limits_{t\in[0,\infty)} |e(t)| < \infty$ or $\int\limits_{t_0}^{+\infty} |e(t)|\,dt < \infty$.

In particular, these requirements are fulfilled for voltages of the form

$$e(t) = b(t+a)^{-n},\ e(t) = be^{-at},\ e(t) = be^{-\frac{(t-a)^2}{\sigma^2}},\ e(t) = b\sin(\omega t + \theta), \tag{32}$$

where $a > 0,\ b, \sigma, \omega \in \mathbb{R},\ n \in \mathbb{N},\ \theta \in [0, 2\pi]$.

Lagrange stability. The numerical solution

$L = 0.5$ nH, $C = 0.5$ pF, $r = 2$ ohm, $g = 0.2$ ohm^{-1}, $t_0 = 0$, $x_0 = (0,0,0)^T$, $\varphi_0(y) = y^3$, $\varphi(y) = y^3$, $h(y) = y^3$, $\psi(y) = y^3$, $e(t) = 100\,e^{-t}\sin(5\,t)$

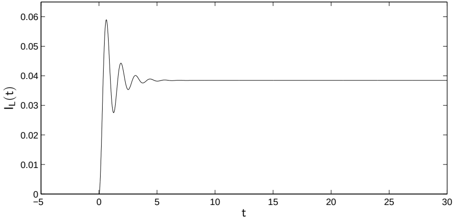

Fig. 10. The current $I_L(t)$

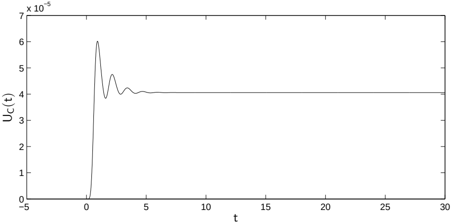

Fig. 11. The voltage $U_C(t)$

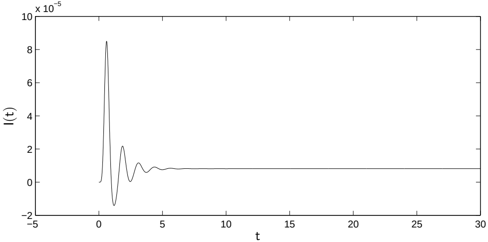

Fig. 12. The current $I(t)$

$L = 0.3$ nH, $C = 0.5$ pF, $r = 2.6$ ohm, $g = 0.2$ ohm^{-1}, $t_0 = 0$, $x_0 = (\pi/6, 0.5, 0)^T$,
$\varphi_0(y) = y^3$, $\varphi(y) = \sin y$, $\psi(y) = \sin y$, $h(y) = \sin y$, $e(t) = 200\sin(0.5t) - 0.2$

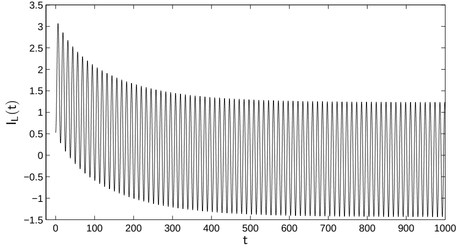

Fig. 13. The current $I_L(t)$

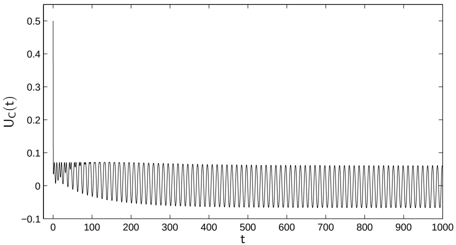

Fig. 14. The voltage $U_C(t)$

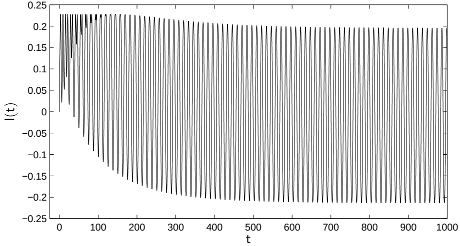

Fig. 15. The current $I(t)$

$L = 50$ pH, $C = 1$ pF, $r = 0.001$ ohm, $g = 1$ ohm^{-1}, $t_0 = 0$, $x_0 = (0,0,0)^T$
$\varphi_0(y) = y^3$, $\varphi(y) = y^3$, $\psi(y) = y^3$, $h(y) = 0.01\,y^3$, $e(t) = 2\sin t$

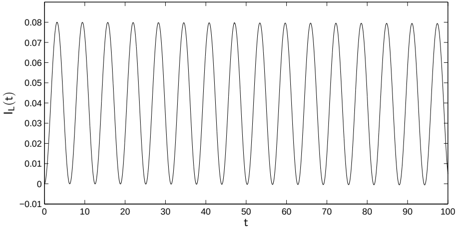

Fig. 16. The current $I_L(t)$

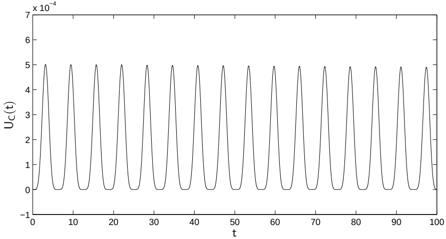

Fig. 17. The voltage $U_C(t)$

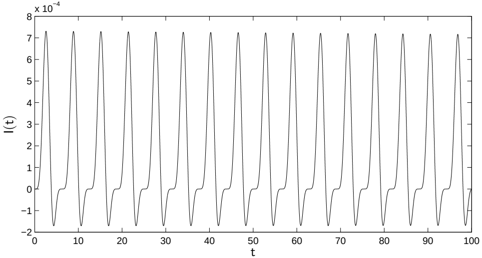

Fig. 18. The current $I(t)$

Consider the system (25)–(27) with the nonlinear resistances and conductance

$$\varphi_0(y) = -y^2,\ \varphi(y) = y^3,\ \psi(y) = y^3,\ h(y) = y^2. \tag{33}$$

It is assumed that there exists $M_e = \sup_{t\in[t_0,\infty)} |e(t)| < +\infty$. Choose

$$\Omega = \Big\{ (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 > m_1, m_1 = \max\Big\{ 1+\sqrt{M_e}, \sqrt[3]{g+r^{-1}}, 3CL^{-1}, \sqrt{\max\{3^{-1}(L(rC)^{-1}-r), 0\}} \Big\}, x_2 < -rx_1 - x_1^3 - m_2, m_2 = \max\{g - 2CL^{-1}r, 0\} \Big\}. \tag{34}$$

Then by Theorem 3 for any initial moment t_0 and any initial currents and voltage $I_L(t_0)$, $U_C(t_0)$, $I(t_0)$ satisfying $U_C(t_0) + rI(t_0) = \psi(I_L(t_0) - I(t_0)) - \varphi(I(t_0))$ and such that $(I_L(t_0), U_C(t_0))^T \in \Omega$ there exists a unique distribution of the currents and voltages in the circuit Fig. 9 only for $t_0 \le t < T$ ($[t_0, T)$ is some finite interval) and the currents and voltages are unbounded.

It means that there exists a unique solution of the Cauchy problem for the DAE (28) with the functions (33), $e(t)$ such that $\sup_{t\in[t_0,\infty)} |e(t)| < +\infty$, and the initial condition $x(t_0) = (I_L(t_0), U_C(t_0), I(t_0))^T$, and this *solution has a finite escape time*.

$L = 10$ pH, $C = 0.5$ pF, $r = 2$ ohm, $g = 0.2$ ohm^{-1}, $x_0 = (2.45, -20.625125, 2.5)^T$
$\varphi_0(x_1) = -x_1^2$, $\varphi(x_3) = x_3^3$, $h(x_2) = x_2^2$, $\psi(x_1 - x_3) = (x_1 - x_3)^3$, $e(t) = 2\sin t$, $t_0 = 0$

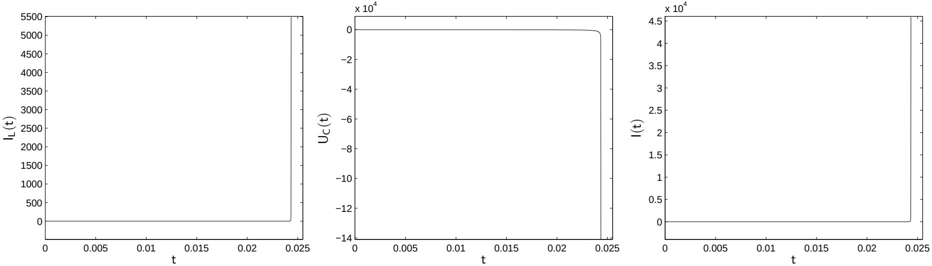

Fig. 19. The current $I_L(t)$ Fig. 20. The voltage $U_C(t)$ Fig. 21. The current $I(t)$

$L = 5$ pH, $C = 0.5$ pF, $r = 2$ ohm, $g = 0.5$ ohm^{-1},
$\varphi_0(x_1) = -x_1^2$, $\varphi(x_3) = x_3^3$, $h(x_2) = x_2^2$, $\psi(x_1 - x_3) = (x_1 - x_3)^3$, $e(t) = 0$,
$t_0 = 0$, $x_0 = (1.1, -4.129, 1.2)^T$

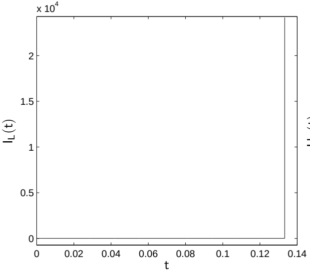

Fig. 22. The current $I_L(t)$

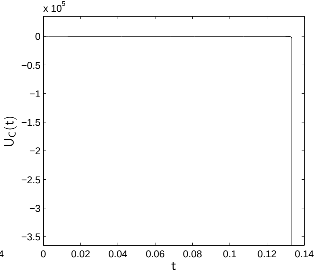

Fig. 23. The voltage $U_C(t)$

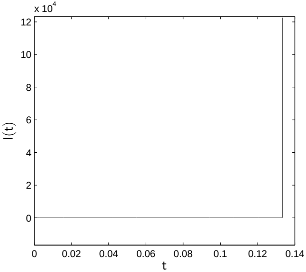

Fig. 24. The current $I(t)$

The block structure of the DAE operator coefficients [3]

Recall that $\lambda A + B$ is **singular** if $n \neq m$ or $n = m$ and $\mathrm{rk}(\lambda A + B) < n$.
There exist the direct decompositions of spaces

$$\mathbb{R}^n = X_s \dot{+} X_r = X_{s_1} \dot{+} X_{s_2} \dot{+} X_r, \quad \mathbb{R}^m = Y_s \dot{+} Y_r = Y_{s_1} \dot{+} Y_{s_2} \dot{+} Y_r \tag{35}$$

such that the **singular operator pencil** $\lambda A + B$ takes the block form

$$\begin{pmatrix} \lambda A_s + B_s & 0 \\ 0 & \lambda A_r + B_r \end{pmatrix}, \quad \begin{array}{l} \lambda A_s + B_s : X_s \to Y_s \text{ is a } \textit{purely singular pencil}, \\ \lambda A_r + B_r : X_r \to Y_r \text{ is a } \textit{regular pencil}. \end{array} \tag{36}$$

We introduce the projectors onto subspaces of the decompositions (35):
$S : \mathbb{R}^n \to X_s$, $F : \mathbb{R}^m \to Y_s$, $S_i : \mathbb{R}^n \to X_{s_i}$, $F_i : \mathbb{R}^m \to Y_{s_i}$, $P : \mathbb{R}^n \to X_r$, $Q : \mathbb{R}^m \to Y_r$.

$$A_s = \begin{pmatrix} A_{gen} & 0 \\ 0 & 0 \end{pmatrix}, B_s = \begin{pmatrix} B_{gen} & B_{und} \\ B_{ov} & 0 \end{pmatrix} : X_{s_1} \dot{+} X_{s_2} \to Y_s = Y_{s_1} \dot{+} Y_{s_2}, \tag{37}$$

$\exists A_{gen}^{-1}$, $A_{gen} = F_1 A S_1|_{X_{s_1}}$, $B_{gen} = F_1 B S_1|_{X_{s_1}}$, $B_{und} = F_1 B S_2|_{X_{s_2}}$, $B_{ov} = F_2 B S_1|_{X_{s_1}}$

- If $\mathrm{rk}(\lambda A + B) = m < n$, the corresponding system of equations is **underdetermined**:

$$A_s = (A_{gen}\, 0), \; B_s = (B_{gen}\, B_{und}) : X_{s_1} \dot{+} X_{s_2} \to Y_s, \quad Y_s = Y_{s_1}, Y_{s_2} = \{0\}. \tag{38}$$

- If $\mathrm{rk}(\lambda A + B) = n < m$, the corresponding system of equations is **overdetermined**:

$$A_s = \begin{pmatrix} A_{gen} \\ 0 \end{pmatrix}, \; B_s = \begin{pmatrix} B_{gen} \\ B_{ov} \end{pmatrix} : X_s \to Y_{s_1} \dot{+} Y_{s_2}, quad X_s = X_{s_1}, X_{s_2} = \{0\}. \tag{39}$$

The case of the singular pencil $\lambda A + B$

It is assumed that $\lambda A_r + B_r$ **is a regular pencil of index 1**: (4).

Then there exist the real spectral projectors of Riss type $\tilde{P}_i : X_r \to X_i$, $\tilde{Q}_i : Y_r \to Y_i$, $i = 1,2$, which decompose spaces X_r, Y_r into direct sums of subspaces

$$X_r = X_1 \dot{+} X_2, \quad Y_r = Y_1 \dot{+} Y_2. \tag{40}$$

$$A_r = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_r = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} : X_1 \dot{+} X_2 \to Y_1 \dot{+} Y_2. \tag{41}$$

By $P_i : \mathbb{R}^n \to X_i$, $Q_i : \mathbb{R}^m \to Y_i$ denote the extensions of the projectors $\tilde{P}_i$, $\tilde{Q}_i$.

To construct the subspaces X_s, Y_s, X_{s_i}, Y_{s_i}, $i = 1, 2$, X_r, Y_r and the corresponding projectors it is necessary to examine solutions of $(\lambda A + B)x = 0$, $(\lambda A^T + B^T)y = 0$.

With respect to the decompositions (35), (40) **any vector** $x \in \mathbb{R}^n$ **can be uniquely represented as the sum**

$$x = x_{s_1} + x_{s_2} + x_1 + x_2, \quad x_{s_i} = S_i x \in X_{s_i},\ x_i = P_i x \in X_i,\ i = 1, 2.$$

The DAE (1) $\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x)$ is equivalent to the system

$$\frac{d}{dt}(A_{gen} x_{s_1}) + B_{gen} x_{s_1} + B_{und} x_{s_2} = F_1 f(t,x), \tag{42}$$

$$B_{ov} x_{s_1} - F_2 f(t,x) = 0, \tag{43}$$

$$\frac{d}{dt}(A_1 x_1) + B_1 x_1 = Q_1 f(t,x), \tag{44}$$

$$Q_2 f(t,x) - B_2 x_2 = 0. \tag{45}$$

The Cauchy problem (1), (2): $\frac{d}{dt}[Ax(t)]+Bx(t)=f(t,x)$, $x(t_0)=x_0$.

Theorem 4. Let $f(t,x)\in C([0,\infty)\times\mathbb{R}^n,\mathbb{R}^m)$, $\frac{\partial}{\partial x}f(t,x)$ be continuous on $[0,\infty)\times\mathbb{R}^n$ and $\lambda A+B$ be a **singular pencil** with the regular component λA_r+B_r of index 1. Suppose

1) $\forall t\geq 0\,\forall x_{s_1}\in X_{s_1}\,\forall x_1\in X_1\,\exists x_{s_2}\in X_{s_2}\,\exists u\in X_2:(t,x_{s_1}+x_{s_2}+x_1+u)\in$ $\in L_0=\{(t,x)\in[0,\infty)\times\mathbb{R}^n \mid (F_2+Q_2)[Bx-f(t,x)]=0\}$;

2) $\forall(t,x_{s_1}+x_{s_2}+x_1+u_i)\in L_0$, $i=1,2$, where $u_i\in X_2$, the operator function $\Phi(u)=\left[\frac{\partial}{\partial x}(Q_2f(t,x_{s_1}+x_{s_2}+x_1+u))-B\right]P_2:X_2\to L(X_2,Y_2)$ is basis invertible on $\operatorname{conv}\{u_1,u_2\}$.

3) there exist self-adjoint positive operators $H_1\in L(X_{s_1})$, $H_2\in L(X_1)$ and for each $T>0$ there exists $R_T>0$ such that

$$(H_1S_1x,A_{gen}^{-1}F_1f(t,x))+(H_2P_1x,A_1^{-1}Q_1f(t,x))\leq 0 \quad \forall(t,x)\in L_0: \\ 0\leq t\leq T,\ \|(S_1+P_1)x\|\geq R_T. \tag{46}$$

Then **for every initial point** $(t_0,x_0)\in L_0$ **there exists a solution** $x(t)$ **of the Cauchy problem** (1), (2) **on** $[t_0,\infty)$**, and if** $\mathrm{rk}(\lambda A+B)=n<m$**, the solution is unique**.

Theorem 5. Let $f(t,x) \in C([0,\infty)\times\mathbb{R}^n,\mathbb{R}^m)$, $\frac{\partial}{\partial x}f(t,x)$ be continuous on $[0,\infty)\times\mathbb{R}^n$, $\lambda A+B$ be a **singular pencil** with the regular component $\lambda A_r + B_r$ of index 1 and the conditions 1), 2) of Theorem 4 be fulfilled. Suppose there exist self-adjoint positive operators $H_1 \in L(X_{s_1})$, $H_2 \in L(X_1)$, number $R>0$ and functions $U(v)\in C((0,\infty),(0,\infty))$, $k(t)\in C([0,\infty),\mathbb{R})$, $\phi_{s_2}(t)\in C([0,\infty),X_{s_2})$ such that $(t,S_1x+\phi_{s_2}(t)+Px)\in L_0$ for all $t\in[0,\infty)$, $v=\frac{1}{2}[(H_1S_1x,S_1x)+(H_2P_1x,P_1x)]$, $\int\limits_c^{+\infty}\frac{dv}{U(v)}=\infty$ $(c>0)$ and $\left(H_1S_1x, A_{gen}^{-1}F_1[-BS_1x-B\phi_{s_2}(t)+f(t,x)]\right)+$ $+\left(H_2P_1x,A_1^{-1}Q_1[-BP_1x+f(t,x)]\right)\le k(t)U(v) \quad \forall (t,x)\in L_0:$ $0\le t\le T$, $\|(S_1+P_1)x\|\ge R_T$.

Then for every initial point $(t_0,x_0)\in L_0$ **there exists a unique solution** $x(t)$ **of the Cauchy problem** (1), (2) **(**$\frac{d}{dt}[Ax(t)]+Bx(t)=f(t,x)$, $x(t_0)=x_0$**) on** $[t_0,\infty)$, for which a choice of the function ϕ_{s_2} with the initial value $\phi_{s_2}(t_0)=S_2x_0$ uniquely defines the component $S_2x(t)=\phi_{s_2}(t)$ when $rk(\lambda A+B)=m<n$.

If, additionally, $\int\limits_{t_0}^{+\infty}k(t)\,dt<\infty$, $\sup\limits_{t\in[0,\infty)}\|\phi_{s_2}(t)\|<\infty$ and $\exists C_1,C_2,M,K>0$ such that $\|Q_2f(t,Sx+P_1x)\|\le C_1$ for all $t\in[0,\infty)$, $\|Sx+P_1x\|\le M$, and $\|F_2f(t,x)\|\le C_2$ for all $t\in[0,\infty)$, $\|x\|\le K$, then for the initial points $(t_0,x_0)\in L_0$ **the equation** (1) **is Lagrange stable** when substituted $S_2x=\phi_{s_2}(t)$, if $rk(\lambda A+B)=n<m$ the component S_2x is absent.

The Lagrange instability of the singular DAE

Theorem 6. Let $f(t,x) \in C([0,\infty) \times \mathbb{R}^n, \mathbb{R}^m)$, $\frac{\partial}{\partial x} f(t,x)$ be continuous on $[0,\infty) \times \mathbb{R}^n$, $\lambda A + B$ be a **singular pencil** with the regular component $\lambda A_r + B_r$ of index 1 and the conditions 1), 2) of Theorem 4 be fulfilled. Let there exist a region $\mathbf{\Omega} \subset X_{s_1} \dot{+} X_1$ such that $(S_1 + P_1)x = 0 \notin \mathbf{\Omega}$ and the component $(S_1 + P_1)x(t)$ of every existing solution $x(t)$ with the initial point $(t_0, x_0) \in L_0$, where $(S_1 + P_1)x_0 \in \mathbf{\Omega}$, remains all the time in $\mathbf{\Omega}$. Suppose there exist self-adjoint positive operators $H_1 \in L(X_{s_1})$, $H_2 \in L(X_1)$ and functions $k(t) \in C([0,\infty), \mathbb{R})$, $\phi_{s_2}(t) \in C([0,\infty), X_{s_2})$, $U(v) \in C((0,\infty),(0,\infty))$ such that $(t, S_1 x + \phi_{s_2}(t) + Px) \in L_0$ for all $t \in [0,\infty)$, $v = \frac{1}{2}[(H_1 S_1 x, S_1 x) + (H_2 P_1 x, P_1 x)]$, $\int\limits_c^{+\infty} \frac{dv}{U(v)} dv < \infty$ $(c > 0)$, $\int\limits_{t_0}^{+\infty} k(t)\, dt = \infty$ and

$$(H_1 S_1 x, A_{gen}^{-1} F_1[-BS_1 x - B\phi_{s_2}(t) + f(t,x)]) + (H_2 P_1 x, A_1^{-1} Q_1[-BP_1 x + f(t,x)]) \geq$$
$$\geq k(t)\, U(v) \quad \forall (t,x) \in L_0 : (S_1 + P_1)x \in \mathbf{\Omega}.$$

Then **for every initial point** $(t_0, x_0) \in L_0$**, where** $(S_1 + P_1)x_0 \in \mathbf{\Omega}$**, there exists a unique solution** $x(t)$ **of the Cauchy problem** (1), (2) $\left(\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x),\ x(t_0) = x_0\right)$, for which a choice of the function ϕ_{s_2} with the initial value $\phi_{s_2}(t_0) = S_2 x_0$ uniquely defines the component $S_2 x(t) = \phi_{s_2}(t)$ when $rk(\lambda A + B) = m < n$, and **this solution has a finite escape time**.

To uniquely define the internal state of the quadripole electric circuit it is necessary to know two input parameters.

In given case, **the investigation of a mathematical model in the conditions of incomplete data** is make, because only one input parameter (the current $\mathrm{I(t)}$) is given.

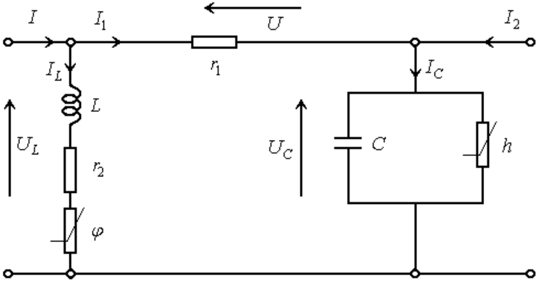

Fig. 25. The electric circuit diagram of the quadripole

An input current $\mathrm{I(t)} \in \mathrm{C}([0,\infty),\mathbb{R})$, a nonlinear resistance $\varphi(\mathrm{y}) \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$, a nonlinear conductance $\mathrm{h(y)} \in \mathrm{C}^1(\mathbb{R},\mathbb{R})$, linear resistances r_1, r_2, an inductance L and a capacitance C are given, r_1, r_2, L, $\mathrm{C} > 0$.

The model of the circuit Fig. 25 [4] is described by the **underdetermined** system of equations with the variables
$x_1 = I_L$, $x_2 = U_C$, $x_3 = I_1$, $x_4 = I_2$:

$$L\frac{dx_1}{dt} + r_2 x_1 - x_2 - r_1 x_3 = -\varphi(x_1), \tag{47}$$

$$C\frac{dx_2}{dt} - x_3 - x_4 = -h(x_2), \tag{48}$$

$$x_1 + x_3 = I(t). \tag{49}$$

The vector form of the system is the DAE

$$\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x), \tag{50}$$

$$A = \begin{pmatrix} L & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} r_2 & -1 & -r_1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

$$f(t,x) = \begin{pmatrix} -\varphi(x_1) \\ -h(x_2) \\ I(t) \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4.$$

$\lambda A + B$ — is a **singular pencil**.

Let there exist number $R>0$ and functions $k(t)\in C([0,\infty),\mathbb{R})$, $U(v)\in C((0,\infty),(0,\infty))$ such that $v=\frac{1}{2}(Lx_1^2+Cx_2^2)$, $\int\limits_c^{+\infty}\frac{dv}{U(v)}=\infty$ and for any $t\in[0,\infty)$, $\sqrt{3x_1^2+x_2^2}\geq R$ the following condition is fulfilled:

$$-(r_2+r_1)x_1^2-x_1\varphi(x_1)-x_2h(x_2)+x_1x_2+\xi(t)x_2+r_1x_1I(t)\leq k(t)U(v).$$

Then by Theorem 5 for any initial values $t_0\geq 0$, $x^0=(I_L(t_0),U_C(t_0),I_1(t_0),I_2(t_0))^T$ satisfying $I_L(t_0)+I_1(t_0)=I(t_0)$ there exists a unique solution $x(t)$ of the equation (50) on $[t_0,\infty)$ with the initial condition $x(t_0)=x^0$, where the component $S_2x(t)=(0,0,0,\xi(t))^T$, $\xi(t)\in C([0,\infty),\mathbb{R})$, is chosen such that $\xi(t_0)=I_1(t_0)+I_2(t_0)$.

If, additionally, $\sup\limits_{t\in[0,\infty)}|I(t)|<\infty$, $\sup\limits_{t\in[0,\infty)}|\xi(t)|<\infty$ and $\int\limits_{t_0}^{+\infty}k(t)\,dt<\infty$, *the solution* $x(t)$ *will be bounded.*

The conditions of Theorem 5 are satisfied for **the particular case:** $\varphi(y)=\alpha_1 y^{2k-1}$, $h(y)=\alpha_2 y^{2r-1}$, where $k,r\in\mathbb{N}$, $\alpha_i>0$, $i=1,2$, $y\in\mathbb{R}$, and $\sup\limits_{t\in[0,\infty)}|I(t)|<\infty$, $\sup\limits_{t\in[0,\infty)}|\xi(t)|<\infty$.

Consider the following **inverse problem** for the two-pole radio engineering filter (the bipole) shown in Fig. 26. Let us verify that by selecting an input current $I = I(t)$ and corresponding initial data it is possible to ensure the evolution of a current I_1 within the bipole so that it will be equal to a given function $I_1 = I_1(t)$, $t_0 \leq t < \infty$.

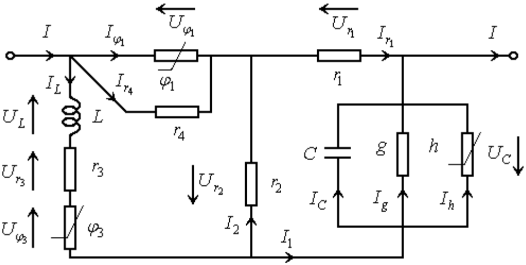

Fig. 26. The electric circuit diagram of the bipole

Here an inductance L, a capacitance C, linear resistances r_k $(k = \overline{1,4})$ and conductivity g are positive real parameters; functions $\varphi_1(y), \varphi_3(y), h(y) \in C^1(\mathbb{R},\mathbb{R})$ characterize nonlinear resistances and conductivity; $I(t)$, $I_1(t) \in C([0,\infty),\mathbb{R})$ are the given currents.

The inverse problem for the two-pole radio engineering filter

The model of the circuit Fig. 26 [5] is described by the **overdetermined** system of equations with the variables $x_1 = I_L$, $x_2 = U_C$, $x_3 = I_{\varphi_1}$

$$L\frac{dx_1}{dt} + (r_2 + r_3)x_1 = r_2 I_1(t) + \varphi_1(x_3) - \varphi_3(x_1), \quad (51)$$

$$C\frac{dx_2}{dt} + g x_2 = I_1(t) - h(x_2), \quad (52)$$

$$x_2 - r_2 x_1 = r_1 I(t) - (r_1 + r_2) I_1(t), \quad (53)$$

$$x_1 + x_3 = I(t) - q\varphi_1(x_3). \quad (54)$$

The vector form of the system is the DAE

$$\frac{d}{dt}[Ax(t)] + Bx(t) = f(t,x), \quad (55)$$

$$A = \begin{pmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} r_2 + r_3 & 0 & 0 \\ 0 & g & 0 \\ -r_2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$f(t,x) = \begin{pmatrix} r_2 I_1(t) + \varphi_1(x_3) - \varphi_3(x_1) \\ I_1(t) - h(x_2) \\ r_1 I(t) - (r_1 + r_2) I_1(t) \\ I(t) - q\varphi_1(x_3) \end{pmatrix}. \quad \mathrm{rk}(\lambda A + B) = 3.$$

$\lambda A + B$ — is a **singular pencil**.

The particular cases.

$$\varphi_1(y)=\alpha_1 y^{2k-1},\varphi_3(y)=\alpha_2 y^{2j-1},h(y)=\alpha_3 y^{2r-1},k,j,r\in\mathbb{N},\alpha_i>0, \tag{56}$$

$$\varphi_1(y)=\alpha_1\sin(y),\varphi_3(y)=\alpha_2\sin(y),h_2(y)=\alpha_3\sin(y),\ \alpha_k>0. \tag{57}$$

By Theorem 5 for any initial values $t_0\ge 0$, $x^0=\big(I_L(t_0),U_C(t_0),I_{\varphi_1}(t_0)\big)^T$ satisfying $\begin{cases} U_C(t_0)-r_2 I_L(t_0)=r_1 I(t_0)-(r_1+r_2)I_1(t_0), \\ I_L(t_0)+I_{\varphi_1}(t_0)=I(t_0)-q\,\varphi_1(I_{\varphi_1}(t_0)) \end{cases}$ there exists a unique solution $x(t)$ of the Cauchy problem for the DAE (55) on $[t_0,\infty)$ with the initial condition $x(t_0)=x^0$ if:

- the nonlinear functions have the form (56), α_1 is sufficiently small and $L\ne C(r_2+r_3)/g$, $k\le j$ or $L=C(r_2+r_3)/g$, $k\le j,r$.
- the nonlinear functions have the form (57) and $\alpha_1<r_4$.

The solution is bounded (in both cases) if $\sup\limits_{t\in[0,\infty)}|I(t)|<\infty$ and $\sup\limits_{t\in[0,\infty)}|I_1(t)|<\infty$.

In particular, these requirements are fulfilled for currents of the form $I(t)=b_1t^{-n_1}$, $I_1(t)=b_2t^{-n_2}$, $I_1(t)=b_1e^{-a_1t}$, $I(t)=b_2e^{-a_2t}$, $I_1(t)=b_1e^{-\frac{(t-a_1)^2}{\sigma_1^2}}$, $I(t)=b_2e^{-\frac{(t-a_2)^2}{\sigma_2^2}}$, $I_1(t)=b_1\sin(\omega_1t+\theta_1)$, $I(t)=b_2\sin(\omega_2t+\theta_2)$, where $n_k\in\mathbb{N}$, $a_k>0$, $b_k,\sigma_k,\omega_k\in\mathbb{R}$, $\theta_k\in[0,2\pi]$, $k=1,2$.

1. M.S. Filipkovska, Extension of solutions of semilinear differential-algebraic equations and applications to nonlinear radiotechnics, Bulletin of V. Karazin Kharkiv National University. Ser. Mathematical Modeling. Information Technology. Automated Control Systems 19 (1015) (2012), 306–319 (in Russian).

2. A.G. Rutkas, M.S. Filipkovska, Extension of solutions of one class of differential-algebraic equations, The Journal of Computational and Applied Mathematics, 1 (2013), 135–145 (in Russian).

3. M.S. Filipkovska, Lagrange stability and numerical method for solving semilinear descriptor equations, Bulletin of V. Karazin Kharkiv National University. Ser. Mathematical Modeling. Information Technology. Automated Control Systems 26 (1156) (2015), 152–167 (in Russian).

4. M. Filipkovskaya, Global solvability of singular semilinear differential equations and applications to nonlinear radio engineering, Challenges of modern technology, 6 (1) (2015), 3–13.

5. M.S. Filipkovska, The global solvability of the overdetermined singular system of differential-algebraic equations and applications of radiotechnics, Radioelectronics and Informatics, 64 (2014), 7–16.

Thank you for your attention!