ON HOMOGENIZATION *for* PERIODIC ELLIPTIC OPERATORS ON AN INFINITE CYLINDER

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INTRODUCTION: BEYOND PURE PERIODICITY

$$-\operatorname{div} A(x_1/\varepsilon, x_2)\nabla u_\varepsilon + u_\varepsilon = f \quad \text{in} \quad \mathbb{R}\times\mathbb{T}$$

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Analytic perturbation theory?

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Analytic perturbation theory?

Before

$$\mathfrak{A}(0) = -\operatorname{div}_x A\nabla_x \quad \text{on} \quad L_2(0,1)$$

$$\mathfrak{N} = \ker(\mathfrak{A}(0)) \simeq \mathbb{C}$$

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Analytic perturbation theory?

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$$\mathfrak{N} = \ker(\mathfrak{A}(0)) \simeq \mathbb{C}$$

Now

$$\mathfrak{A}(0) = -\operatorname{div}_{x_1} A\nabla_{x_1} \quad \text{on} \quad L_2((0,1)\times\mathbb{T})$$

$$\mathfrak{N} = \ker(\mathfrak{A}(0)) \simeq L_2(\mathbb{T})$$

Suslina · 2004

$$A(x) = \begin{pmatrix} A_{11}(x) & 0 \\ 0 & A_{22}(x) \end{pmatrix}$$

$$\Downarrow$$

$$\left\| \left(-\operatorname{div} A(x_1/\varepsilon, x_2)\nabla + 1\right)^{-1} - \left(-\operatorname{div} A^0(x_2)\nabla + 1\right)^{-1} \right\|_{\mathbf{B}(L_2)} \leqslant C\varepsilon$$

PROBLEM FORMULATION

d_1 the number of **periodic** directions

d_2 the number of **non-periodic** directions

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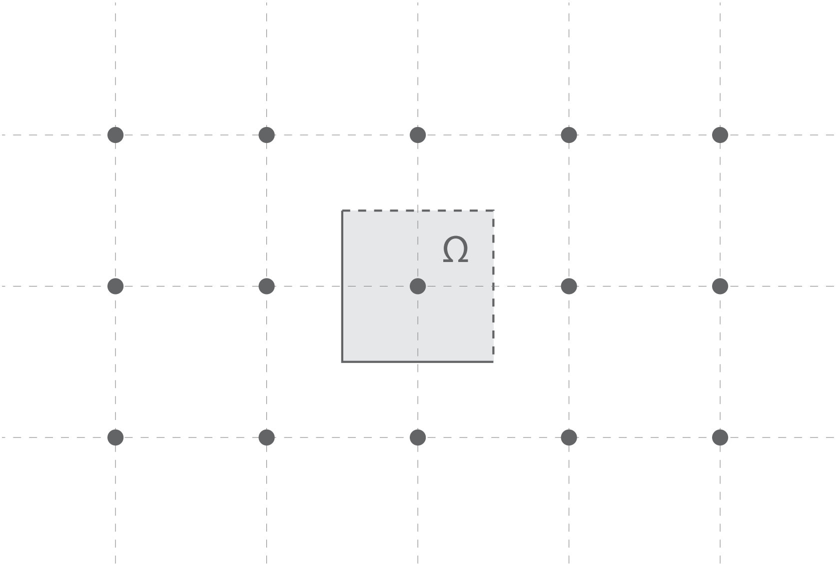
d_2 the number of **non-periodic** directions

Λ a lattice in $\mathbb{R}^{d_1}$ with basic cell Ω

Example: $\Lambda = \mathbb{Z}^{d_1}$, $\Omega = [-1/2, 1/2)^{d_1}$



d_1	the number of **periodic** directions
d_2	the number of **non-periodic** directions
Λ	a lattice in $\mathbb{R}^{d_1}$ with basic cell Ω
Ξ	the cylinder $\mathbb{R}^{d_1} \times \mathbb{T}^{d_2}$

Definition of the Operator

$$\mathcal{A}^{\varepsilon} = D^{*} A^{\varepsilon} D \colon H^{1}(\Xi)^{n} \to H^{-1}(\Xi)^{n}$$

- $\varepsilon \in \mathscr{E} = (0, \varepsilon_0]$
- $D = -i\nabla$
- $A^{\varepsilon}(x) = A(x_1/\varepsilon, x_2)$
- $A \colon \Xi \to \mathbf{B}(\mathbb{C}^{d \times n})$

Assumptions

$$\mathcal{A}^{\varepsilon} = D^{*} A^{\varepsilon} D \colon H^{1}(\Xi)^{n} \to H^{-1}(\Xi)^{n}$$

1. A is periodic w.r.t. Λ
2. $A \in \mathrm{Lip}(\mathbb{T}^{d_2}; L_{\infty}(\Omega))$
3. For any $u \in H^{1}(\Xi)^{n}$

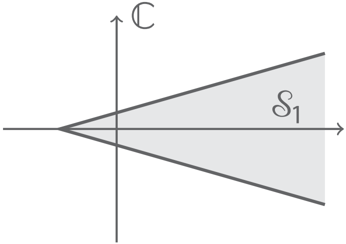
$$\mathrm{Re}(A^{\varepsilon} Du, Du)_{2,\Xi} + C_A \|u\|_{2,\Xi}^{2} \geqslant c_A \|Du\|_{2,\Xi}^{2}$$

uniformly in $\varepsilon \in \mathcal{E}$

$\mathcal{A}^\varepsilon$ strictly *m*-sectorial

$\mathcal{A}^\varepsilon - \mu \colon H^1(\Xi)^n \to H^{-1}(\Xi)^n$

isomorphism $\forall \mu \notin \mathcal{S}_1$



Goal

$$(\mathcal{A}^{\varepsilon} - \mu)^{-1} \simeq ?_1$$

1

$$D(\mathcal{A}^{\varepsilon} - \mu)^{-1} \simeq ?_2$$

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1

$$(\mathcal{A}^{\varepsilon} - \mu)^{-1} \simeq ?_1$$

$$D(\mathcal{A}^{\varepsilon} - \mu)^{-1} \simeq ?_2$$

2

$$\left\|(\mathcal{A}^{\varepsilon} - \mu)^{-1} - ?_1\right\|_{\mathbf{B}(L_2(\Xi))} \leqslant ?$$

$$\left\|D(\mathcal{A}^{\varepsilon} - \mu)^{-1} - ?_2\right\|_{\mathbf{B}(L_2(\Xi))} \leqslant ?$$

Overview of the Results

$$\left\|(\mathcal{A}^{\varepsilon}-\mu)^{-1}-(\mathcal{A}^{0}-\mu)^{-1}\right\|_{\mathbf{B}(L_2(\Xi))}\leqslant C\varepsilon$$

$$\left\|D_2(\mathcal{A}^{\varepsilon}-\mu)^{-1}-D_2(\mathcal{A}^{0}-\mu)^{-1}\right\|_{\mathbf{B}(L_2(\Xi))}\leqslant C\varepsilon$$

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$$\left\|D_1(\mathcal{A}^{\varepsilon}-\mu)^{-1}-D_1(\mathcal{A}^{0}-\mu)^{-1}-\varepsilon D_1\mathcal{K}_{\mu}^{\varepsilon}\right\|_{\mathbf{B}(L_2(\Xi))}\leqslant C\varepsilon$$

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$$\left\|(\mathcal{A}^{\varepsilon}-\mu)^{-1}-(\mathcal{A}^{0}-\mu)^{-1}-\varepsilon\mathcal{C}_{\mu}^{\varepsilon}\right\|_{\mathbf{B}(L_2(\Xi))}\leqslant C\varepsilon^{2}$$

EFFECTIVE OPERATOR AND CORRECTORS

Cell Problem

$N\colon \Xi \to \mathbf{B}(\mathbb{C}^{n\times d}, \mathbb{C}^n)$

the periodic solution of the elliptic problem

$$\begin{pmatrix} D_1 \\ 0 \end{pmatrix}^* A(\cdot, x_2)\left(\begin{pmatrix} D_1 \\ 0 \end{pmatrix} N(\cdot, x_2) + I\right) = 0, \qquad \int_\Omega N(\cdot, x_2) = 0$$

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$$\Downarrow$$

$$N \in \mathrm{Lip}(\mathbb{T}^{d_2}; \tilde{H}^{1}(\Omega))$$

Effective Operator

$$\mathcal{A}^0 = D^* A^0 D \colon H^1(\Xi)^n \to H^{-1}(\Xi)^n$$

$$A^0(x_2) = |\Omega|^{-1} \int_\Omega A(y_1, x_2) \left(\begin{pmatrix} D_1 \\ 0 \end{pmatrix} N(y_1, x_2) + I \right) dy_1$$

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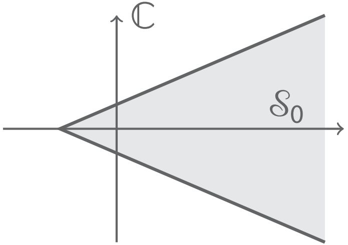
$$A, N \in \mathrm{Lip} \;\Rightarrow\; A^0 \in \mathrm{Lip}$$

$\mathcal{A}^0$ strictly *m*-sectorial

$$\mathcal{A}^0 - \mu \colon H^1(\Xi)^n \to H^{-1}(\Xi)^n$$

isomorphism $\forall \mu \notin \mathcal{S}_0$



Corrector $\mathcal{K}_\mu^\varepsilon$

$$\mathcal{K}_\mu^\varepsilon = N^\varepsilon D(\mathcal{A}^0 - \mu)^{-1}\mathcal{P}^\varepsilon\colon L_2(\Xi)^n \to H^1(\Xi)^n$$

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$$\mathcal{P}^\varepsilon \sim \mathbb{1}_{\varepsilon^{-1}\Omega^*}$$

Ω^* the first Brillouin zone of Λ

Corrector $\mathcal{K}_\mu^\varepsilon$

$$\mathcal{K}_\mu^\varepsilon = N^\varepsilon D(\mathcal{A}^0 - \mu)^{-1}\mathcal{P}^\varepsilon \colon L_2(\Xi)^n \to H^1(\Xi)^n$$

$$\mathcal{P}^\varepsilon = (\mathcal{F} \otimes \mathcal{I})^{-1} \mathbb{1}_{\varepsilon^{-1}\Omega^*} (\mathcal{F} \otimes \mathcal{I})$$

Ω^* the first Brillouin zone of Λ

$\mathcal{F}$ the Fourier transform in the x_1-variable

Corrector $\mathcal{C}_\mu^\varepsilon$

$$\mathcal{C}_\mu^\varepsilon = \left(\mathcal{K}_\mu^\varepsilon - \mathcal{L}_\mu\right) + \left((\mathcal{K}_\mu^\varepsilon)^+ - \mathcal{L}_\mu^+\right)^* \colon L_2(\Xi)^n \to L_2(\Xi)^n$$

Corrector $\mathcal{C}^{\varepsilon}_{\mu}$

$$\mathcal{C}^{\varepsilon}_{\mu} = \left(\mathcal{K}^{\varepsilon}_{\mu} - \mathcal{L}_{\mu}\right) + \left((\mathcal{K}^{\varepsilon}_{\mu})^{+} - \mathcal{L}^{+}_{\mu}\right)^{*} \colon L_2(\Xi)^n \to L_2(\Xi)^n$$

$$\mathcal{L}_{\mu} = (\mathcal{A}^0 - \mu)^{-1} D^{*} \mathcal{L}\, D (\mathcal{A}^0 - \mu)^{-1}$$

$\mathcal{L} \colon H^1(\Xi)^n \to L_2(\Xi)^n$

first-order differential operator with coefficients depending on x_2

Corrector $\mathcal{C}_\mu^\varepsilon$

$$\mathcal{C}_\mu^\varepsilon = \left(\mathcal{K}_\mu^\varepsilon - \mathcal{L}_\mu\right) + \left(\left(\mathcal{K}_\mu^\varepsilon\right)^+ - \mathcal{L}_\mu^+\right)^*: L_2(\Xi)^n \to L_2(\Xi)^n$$

$$\mathcal{L}_\mu = (\mathcal{A}^0 - \mu)^{-1} D^* \mathcal{L}\, D(\mathcal{A}^0 - \mu)^{-1}$$

$$\mathcal{L} \sim k \mapsto |\Omega|^{-1} \int_\Omega \left(\left(\begin{matrix} k \\ D_2 \end{matrix} \right) N^+(y_1, \cdot) \right)^* A(y_1, \cdot) \left(I + \left(\begin{matrix} D_1 \\ 0 \end{matrix} \right) N(y_1, \cdot) \right) dy_1$$

MAIN RESULTS

Theorem

$\forall \mu \notin \mathcal{S}_1 \cup \mathcal{S}_0 \quad \forall \varepsilon \in \mathcal{E}$

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} - (\mathcal{A}^0 - \mu)^{-1}\|_{\mathbf{B}(L_2(\Xi))} \leqslant C\varepsilon$$

$$\|D_2(\mathcal{A}^\varepsilon - \mu)^{-1} - D_2(\mathcal{A}^0 - \mu)^{-1}\|_{\mathbf{B}(L_2(\Xi))} \leqslant C\varepsilon$$

$$\|D_1(\mathcal{A}^\varepsilon - \mu)^{-1} - D_1(\mathcal{A}^0 - \mu)^{-1} - \varepsilon D_1 \mathcal{K}_\mu^\varepsilon\|_{\mathbf{B}(L_2(\Xi))} \leqslant C\varepsilon$$

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} - (\mathcal{A}^0 - \mu)^{-1} - \varepsilon \mathcal{C}_\mu^\varepsilon\|_{\mathbf{B}(L_2(\Xi))} \leqslant C\varepsilon^2$$

Sharp order in ε

C **depends** only on n, d_1, d_2, μ, Λ, c_A, C_A and $\|A\|_{C^{0,1}}$

SCHEME OF THE PROOF

The Scaling Transformation and the Floquet–Bloch Theory

$$(\mathcal{A}^{\varepsilon} - \mu)^{-1} \simeq \varepsilon^{2} \int_{\Omega^*}^{\oplus} (\mathcal{A}(k, \varepsilon) - \varepsilon^{2}\mu)^{-1} dk$$

$$\mathcal{A}(k, \varepsilon) \colon \tilde{H}^{1}(\Omega \times \mathbb{T}^{d_2}) \to \tilde{H}^{-1}(\Omega \times \mathbb{T}^{d_2})$$

$$\mathcal{A}(k, \varepsilon) = \varepsilon^{2} \begin{pmatrix} \varepsilon^{-1}(D_1 + k) \\ D_2 \end{pmatrix}^{*} A \begin{pmatrix} \varepsilon^{-1}(D_1 + k) \\ D_2 \end{pmatrix}$$

The Scaling Transformation and the Floquet–Bloch Theory

$$(\mathcal{A}^\varepsilon - \mu)^{-1} \simeq \varepsilon^2 \int_{\Omega^*}^{\oplus} \left(\mathcal{A}(k,\varepsilon) - \varepsilon^2 \mu\right)^{-1} dk$$

$$(\mathcal{A}^0 - \mu)^{-1} \simeq \varepsilon^2 \int_{\Omega^*}^{\oplus} \left(\mathcal{A}^0(k,\varepsilon) - \varepsilon^2 \mu\right)^{-1} dk$$

$$\mathcal{K}_\mu^\varepsilon \simeq \varepsilon^2 \int_{\Omega^*}^{\oplus} \mathcal{K}_\mu(k,\varepsilon)\, dk$$

$$\mathcal{C}_\mu^\varepsilon \simeq \varepsilon^2 \int_{\Omega^*}^{\oplus} \mathcal{C}_\mu(k,\varepsilon)\, dk$$

A «Resolvent» Identity

$$\left(\mathcal{A}(k,\varepsilon)-\varepsilon^2\mu\right)^{-1}-\left(\mathcal{A}^0(k,\varepsilon)-\varepsilon^2\mu\right)^{-1}-\mathcal{K}_\mu(k,\varepsilon)$$
$$=\left(\mathcal{A}(k,\varepsilon)-\varepsilon^2\mu\right)^{-1}\left(\ldots\right)\left(\mathcal{A}^0(k,\varepsilon)-\varepsilon^2\mu\right)^{-1}$$

BEYOND
PURE PERIODICITY *and*
EVEN FURTHER

Locally Periodic Operators

$$\mathcal{A}^\varepsilon = D^* A^\varepsilon D \colon H^1(\mathbb{R}^d)^n \to H^{-1}(\mathbb{R}^d)^n$$

- $A^\varepsilon(x) = A(x, x/\varepsilon)$
- $A \colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbf{B}(\mathbb{C}^{d \times n})$
- $A \in C^{0,1}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(\Omega))$

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} - (\mathcal{A}^0 - \mu)^{-1}\|_{\mathbf{B}(L_2(\mathbb{R}^d))} \leqslant C\varepsilon$$

$$\|D(\mathcal{A}^\varepsilon - \mu)^{-1} - D(\mathcal{A}^0 - \mu)^{-1} - \varepsilon D\mathcal{K}_\mu^\varepsilon\|_{\mathbf{B}(L_2(\mathbb{R}^d))} \leqslant C\varepsilon$$

$$\|(\mathcal{A}^\varepsilon - \mu)^{-1} - (\mathcal{A}^0 - \mu)^{-1} - \varepsilon\mathcal{C}_\mu^\varepsilon\|_{\mathbf{B}(L_2(\mathbb{R}^d))} \leqslant C\varepsilon^2$$

References

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