The Maxwell system in waveguides with non-homogeneous anisotropic filling

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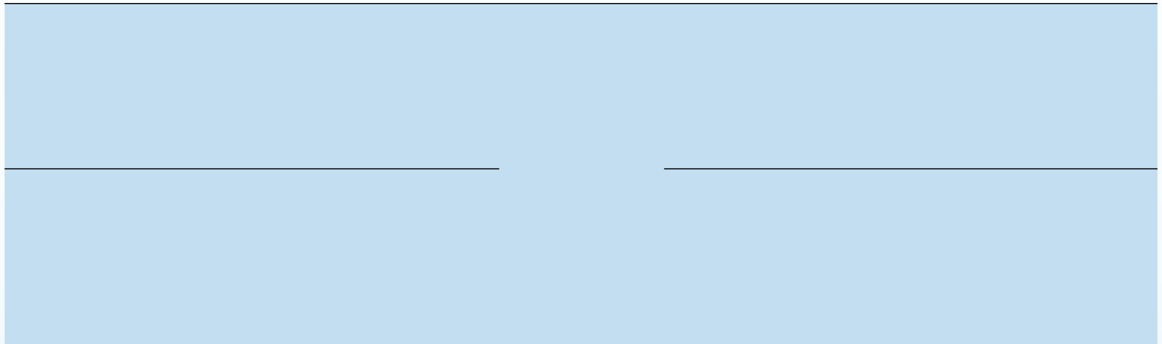
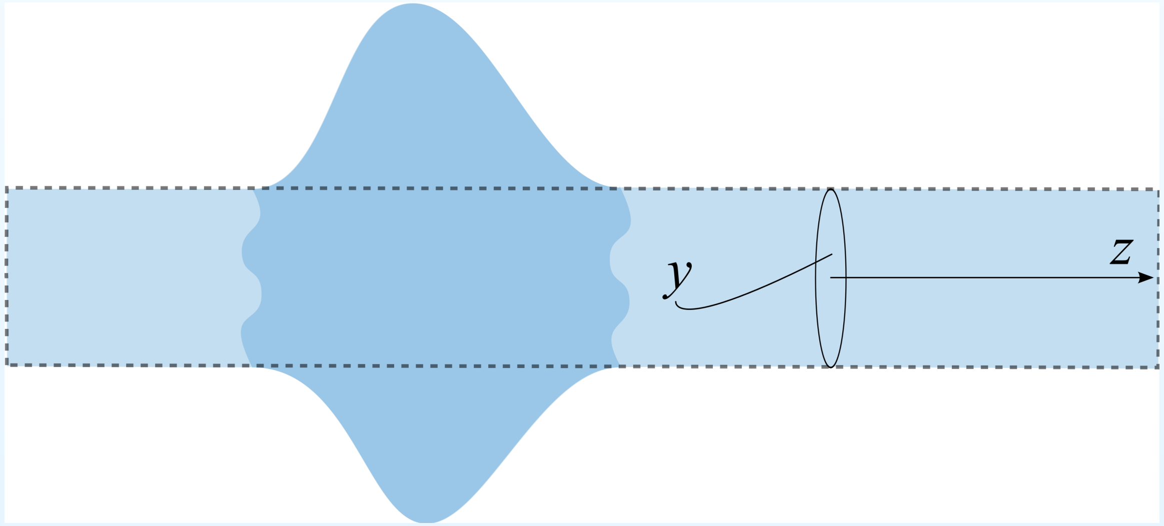
EM waveguides. State of the art



[V'66], [ML'71], [NF'72]
Wiener-Hopf
Mode matching



[IKS'91], [GI'13]
$\varepsilon, \mu = \text{const} \cdot I$
[BDS'99], [BDS'00], [D'07]
$\varepsilon(y), \mu(y)$
block-diagonal

An actual problem is to extend the class of electromagnetic waveguides admitting a mathematically accurate investigation: geometry and filling medium.

The Maxwell system

- The stationary Maxwell system

$$i\varepsilon(x)^{-1}\mathrm{rot}u^2(x) - ku^1(x) = f(x), \ -i\mathrm{div}(\mu(x)u^2(x)) = 0,$$
$$-i\mu(x)^{-1}\mathrm{rot}u^1(x) - ku^2(x) = 0, \quad i\mathrm{div}(\varepsilon(x)u^1(x)) = h(x). \quad (1)$$

Boundary conditions

$$u^1_\tau(x) = 0, \ (\mu u^2)_\nu(x) = 0. \quad (2)$$

u^1 and u^2 are electric and magnetic vectors;
ε and μ are dielectric and magnetic permittivity matrices.

- The system (1) is over-determined. Compatibility condition (charge conservation law)

$$\mathrm{div}\,(\varepsilon(x)f(x)) - ikh(x) = 0. \quad (3)$$

Elliptization

- Augmented Maxwell system, [O'56],[GK'74], [P'84], [BS'90].

$$i\varepsilon^{-1}\mathrm{rot}u^2+i\nabla a^2-ku^1=f^1,\ -i\mathrm{div}(\mu u^2)-ka^1=h^1,$$
$$-i\mu^{-1}\mathrm{rot}u^1-i\nabla a^1-ku^2=f^2,\quad i\mathrm{div}(\varepsilon u^1)-ka^2=h^2 \tag{4}$$

  with boundary conditions

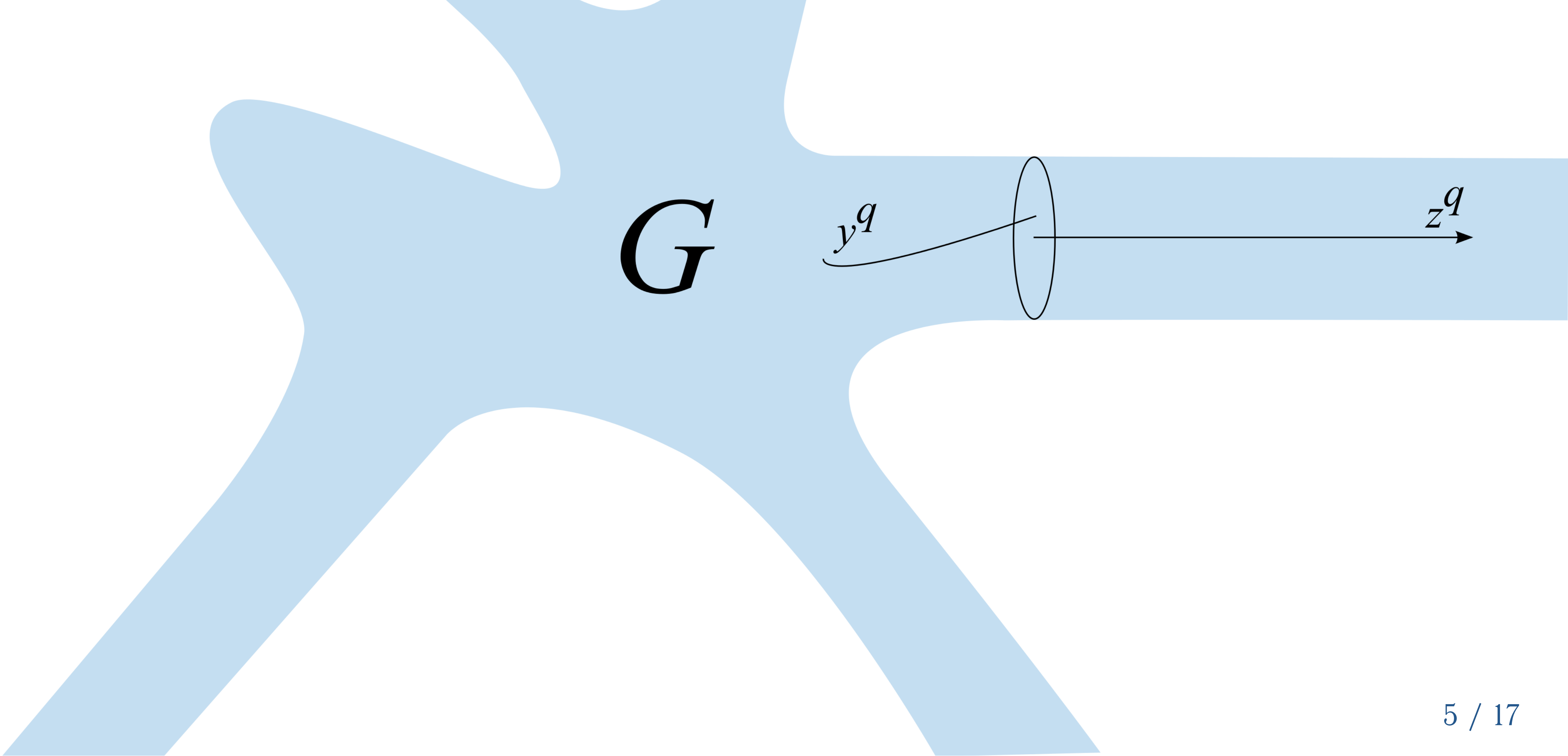
$$-u^1_{\tau_2}=g^1,\quad u^1_{\tau_1}=g^2,\quad (\mu u^2)_\nu=g^3,\quad a^2=g^4 \tag{5}$$

  is elliptic and self-adjoint with respect to a Green formula.
- Theory of self-adjoint problems for elliptic systems in domains with several cylindrical ends. Such a theory was (firstly) developed in [NP'91].
- "Return" to the original Maxwell system.

For empty waveguide ($\varepsilon=\mu=I$) with several cylindrical ends the plan was implemented in [PP'14].

Waveguide

$G \subset \mathbb{R}^3$, coinciding outside a large ball with the union of non-overlapping semicylinders $G \cap \Pi_+^q = \{(y^q, z^q) : y^q \in \Omega^q, z^q > 0\}$, $q = 1, \ldots, \mathcal{T} < \infty$.



Permittivity matrices ε and μ

- Dielectric and magnetic permittivity

$$\overline{G} \ni x \mapsto \varepsilon(x),\ \mu(x)$$

  are **positive-definite** 3×3 **matrix** valued smooth **functions**.

- In every cylindrical end $G \cap \Pi^q_+ = \{(y^q, z^q) : y^q \in \Omega^q, z^q > 0\}$ the matrices $\varepsilon(y^q, z^q)$ and $\mu(y^q, z^q)$ converge as $z^q \to +\infty$ to **positive-definite** 3×3 **matrix** valued smooth **functions**

$$\overline{\Omega^q} \ni y^q \mapsto \varepsilon^q(y^q),\ \mu^q(y^q).$$

- Exponential convergence rate. For a $\delta > 0$ the estimates

$$|\varepsilon(y^q, z^q) - \varepsilon^q(y^q)| + |\nabla(\varepsilon(y^q, z^q) - \varepsilon^q(y^q))| = O(\exp(-\delta z^q)),$$
$$|\mu(y^q, z^q) - \mu^q(y^q)| + |\nabla(\mu(y^q, z^q) - \mu^q(y^q))| = O(\exp(-\delta z^q)),$$

  hold as $z^q \to +\infty$, uniformly with respect to $y^q \in \overline{\Omega^q}$.

- No other restrictions are imposed on the matrices ε and μ.

Continuous and point spectrum

- A solution U to the homogeneous problem (1), (2): $U(x) \leq \mathrm{Const}\,(|x|+1)^N$, $U \notin L_2(G)$ is by definition a continuous spectrum eigenfunction (CSE). The number k belongs to the continuous spectrum of the problem (1), (2). Denote by $E_c(k)$ the linear hull of CSEs.
- A solution $U \in L_2(G)$ to the homogeneous problem (1), (2) is by definition an eigenfunction. The corresponding number k is an eigenvalue. Denote by $E_p(k)$ the eigenspace. The eigenvalues are isolated and have finite multiplicities. The set of eigenvalues is called the point spectrum.
- To introduce the scattering matrix we are to choose a basis in the space $E_c(k)$ with elements, having a specific asymptotics.
- Such an asymptotics is described in terms of incoming and outgoing waves.

Incoming and outgoing waves

- Consider the model problem of the form (1), (2) in a cylinder $\Pi^q = \Omega^q \times \mathbb{R}$ with matrices $\varepsilon^q(y^q)$ and $\mu^q(y^q)$.
- Waves are solutions to the model problem of the form

$$\exp(i\lambda z^q) \sum_{r=0}^{\varkappa-1} (iz^q)^r \varphi^{(\varkappa-1-r)}(y^q), \qquad (6)$$

with a real λ and $\varkappa \geq 1$. A solution (6) with $\varkappa > 1$ may exist only for isolated "threshold" values of k.

Proposition 1. *The space W^q, spanned by solutions of the form* (6)*, has an even dimension $2\varsigma^q$. There exists a basis in W^q, consisting of ς^q "incoming" and ς^q "outgoing" waves.*

- Incoming (outgoing) waves bring energy from $+\infty$ (to $+\infty$).

Waves in G

- Let $\eta \in C^\infty(\mathbb{R})$ be a smooth cut-off function such that $0 \le \eta(t) \le 1$ with $t \in \mathbb{R}$, $\eta(t) = 0$ with $t < 0$, $\eta(t) = 1$ with $t > 1$.
- For every wave $w \in W^q$ introduce a function

$$G \cap \Pi_+^q \ni (y^q, z^q) \mapsto \eta(z^q - T)w(y^q, z^q),$$

  and extend it by zero to the domain G. All functions constructed by this procedure are called waves in G.
- The waves in G, corresponding to basis incoming (outgoing) waves in the spaces $W^1, \dots, W^{\mathcal{T}}$, we call incoming (outgoing), enumerate with a single index, and denote by $u_1^+, \dots, u_\Upsilon^+$ $(u_1^-, \dots, u_\Upsilon^-)$.

Scattering matrix (when k is not an eigenvalue)

Theorem 2. *Let k belong to the continuous spectrum of the problem (1), (2), and k be not an eigenvalue. Then in the space $E_c(k)$ of contionuous spectrum eigenfunctions there exists a basis $Y_1^+, \dots, Y_\Upsilon^+$ with an asymptotics*

$$Y_j^+(x) = u_j^+ + \sum_{l=1}^{\Upsilon} s_{jl} u_l^- + O(e^{-\alpha|x|}), \quad j = 1, \dots, \Upsilon, \tag{7}$$

as $|x| \to \infty$, where $\alpha > 0$ is a sufficiently small number. The matrix s with elements s_{jl} is unitary.

The matrix s, introduced in the Theorem 2, is called the scattering matrix of the problem (1), (2).

Scattering matrix

Theorem 3. *Let k belong to the continuous spectrum and be an eigenvalue of the problem* (1), (2) *(obviously, $E_p(k) \subset E_c(k)$). Then in the quotient space $E_c(k)/E_p(k)$ there exists a basis with representatives $Y_1^+, \dots, Y_\Upsilon^+$, subject to an asymptotics*

$$Y_j^+(x) = u_j^+ + \sum_{l=1}^{\Upsilon} s_{jl} u_l^- + O(e^{-\alpha|x|}), \quad j = 1, \dots, \Upsilon,$$

as $|x| \to \infty$, where $\alpha > 0$ is a sufficiently small number. The matrix s with elements s_{jl} does not depend on the choice of the representatives and is unitary.

The matrix s, introduced in the Theorem 3, is called the scattering matrix of the problem (1), (2).

The weighted Sobolev space

Introduce a positive function $\rho_\alpha \in C^\infty(\overline{G})$:

$$\rho_\alpha(y^q, z^q) = \exp(\alpha z^q), \quad (y^q, z^q) \in G \cap \Pi_+^q,$$

for $q = 1, \dots, \mathcal{T}$, with the number α from **(7)**.
Denote by $H_\alpha^l(G)$, $l \geq 0$, the closure of $C_c^\infty(\overline{G})$ in the norm

$$\|u; H_\alpha^l(G)\| := \|\rho_\alpha u; H^l(G)\| = \Big(\sum_{|\sigma|=0}^{l} \int_G |D^\sigma(\rho_\alpha u)|^2 \, dx \Big)^{1/2}.$$

The space of vector valued functions with d components in $H_\alpha^l(G)$ is denoted by $H_\alpha^l(G; \mathbb{C}^d)$

The radiation principle (when k is not an eigenvalue)

Theorem 4. *Suppose k is not an eigenvalue of the problem* (1), (2). *Let $f \in H^l_\alpha(G;\mathbb{C}^3)$, $h \in H^l_\alpha(G;\mathbb{C})$ satisfy compatibility condition* (3). *Then there exists a unique solution $U = (u^1, u^2)$ to the problem* (1), (2), *subject to the radiation conditions*

$$V := U - \sum_{j=1}^{\Upsilon} c_j u_j^- \in H_\alpha^{l+1}(G;\mathbb{C}^6).$$

Here $c_j = i(F, Y_j^-)_G$ with $F := (\varepsilon f, 0)$, $Y_j^- := \sum_{l=1}^{\Upsilon}(s^{-1})_{jl} Y_l^+$, and Y_l^+ from the Theorem 2. The estimate

$$\|V; H_\alpha^{l+1}(G;\mathbb{C}^6)\| + \sum_{j=1}^{\Upsilon} |c_j| \leq \mathrm{const}(\|f; H_\alpha^l(G;\mathbb{C}^3)\| + \|h; H_\alpha^l(G;\mathbb{C})\|)$$

holds.

The radiation principle

Theorem 5. *Let* $Z_1, \ldots, Z_d$ *be a basis of* $E_p(k)$ *an let* $f \in H^l_\alpha(G;\mathbb{C}^3)$, $h \in H^l_\alpha(G;\mathbb{C})$ *satisfy compatibility condition* (3) *and orthogonality conditions* $(F, Z_j)_G = 0$, $j = 1, \ldots, d$, *where* $F := (\varepsilon f, 0)$. *Then there exists a solution* $U = (u^1, u^2)$ *to the problem* (1), (2), *subject to the radiation conditions*

$$V := U - \sum_{j=1}^{\Upsilon} c_j u_j^- \in H^{l+1}_\alpha(G;\mathbb{C}^6).$$

Here $c_j = i(F, Y_j^-)_G$ *with* $Y_j^- := \sum_{l=1}^{\Upsilon}(s^{-1})_{jl} Y_l^+$, *and* Y_l^+ *from the Theorem 3. The solution* U *is defined up to an arbitrary summand in* $E_p(k)$ *and*

$$\|V; H^{l+1}_\alpha(G;\mathbb{C}^6)\| + \sum_{j=1}^{\Upsilon} |c_j| \leq$$
$$\leq \operatorname{const}(\|f; H^l_\alpha(G;\mathbb{C}^3)\| + \|h; H^l_\alpha(G;\mathbb{C})\| + \|\rho_\alpha V; L_2(G;\mathbb{C}^6)\|). \qquad (8)$$

The solution U_0, *satisfying* $(U_0, Z_j)_G = 0$, $j = 1, \ldots, d$, *is unique; for* U_0 *the estimate* (8) *holds with the right-hand side changed by* $\operatorname{const}(\|f; H^l_\alpha(G;\mathbb{C}^3)\| + \|h; H^l_\alpha(G;\mathbb{C})\|)$.

Thank you for your attention

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