Initial Boundary Value Problems for Integrable Nonlinear Equations: a Riemann–Hilbert Approach

Dmitry Shepelsky

Institute for Low Temperature Physics, Kharkiv, Ukraine

Summer School "Spectral Theory, Differential Equations and Probability"
Mainz, 2016

Example: IBVP for focusing NLS

with decaying initial data and (asymptotically) periodic boundary conditions

Let $q(x,t)$ be the solution of the IBV problem for focusing nonlinear Schrödinger equation (NLS):

- $\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \qquad x > 0, t > 0,$
- $q(x,0) = q_0(x)$ fast decaying as $x \to +\infty$
- $q(0,t) = g_0(t)$ time-periodic $\boxed{g_0(t) = \alpha\, \mathrm{e}^{2\mathrm{i}\omega t}}$ $\alpha > 0, \omega \in \mathbb{R}$
  $(q(0,t) - \alpha\, \mathrm{e}^{2\mathrm{i}\omega t} \to 0$ as $t \to +\infty)$

▷ Question: How does $q(x,t)$ behave for large t ?

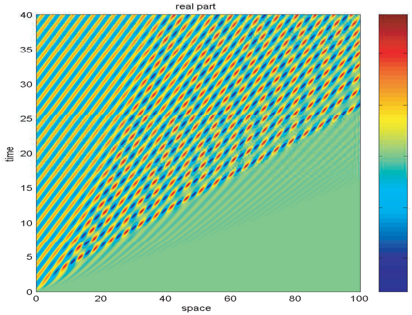
▷ Numerics: Qualitatively different pictures for parameter ranges:

(i) $\omega < -3\alpha^2$

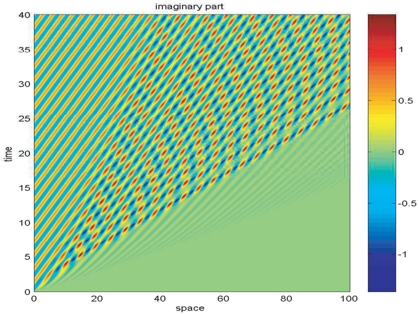
(ii) $-3\alpha^2 < \omega < \frac{\alpha^2}{2}$

(iii) $\omega > \frac{\alpha^2}{2}$

Numerics for $\omega < -3\alpha^2$, I



Real part $\mathrm{Re}\, q(x,t)$



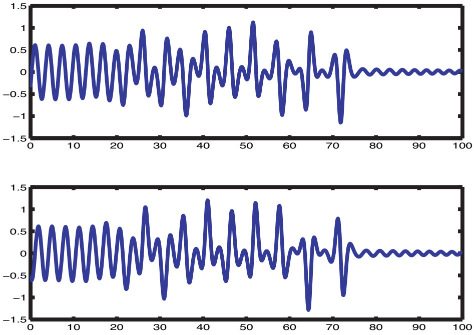
Imaginary part $\mathrm{Im}\, q(x,t)$

$\alpha = \sqrt{3/8}, \;\; \omega = -13/8$

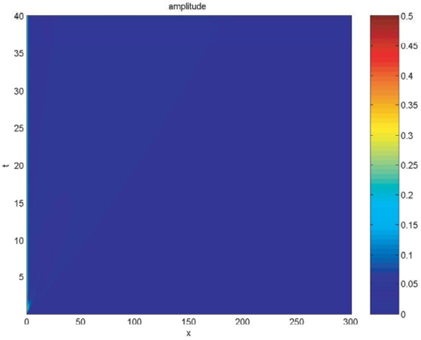
$q_0(x) \equiv 0, \; g_0(t) = \alpha \mathrm{e}^{2\mathrm{i}\omega t} + \mathrm{O}(\mathrm{e}^{-10t^2})$

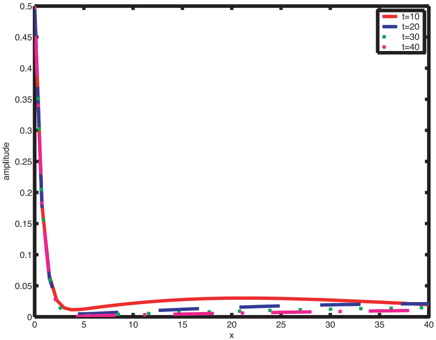
Numerics for $\omega < -3\alpha^2$, II

Numerical solution for $t = 20$, $0 < x < 100$.
Upper: real part $\mathrm{Re}\, q(x, 20)$. Lower: imaginary part $\mathrm{Im}\, q(x, 20)$.



Numerics for $\omega \geq \alpha^2/2$



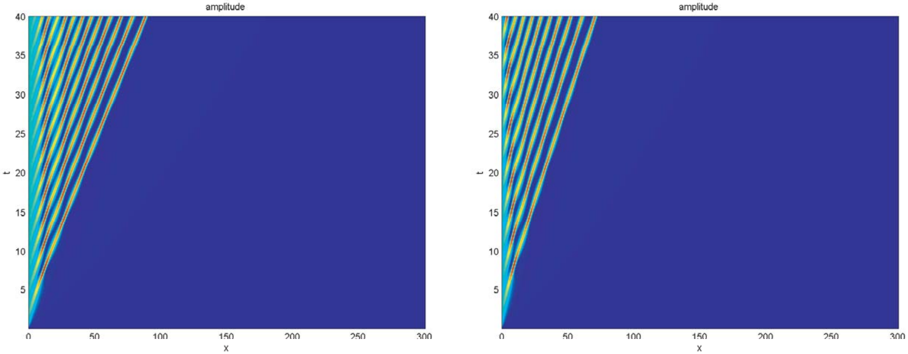


Amplitude of $q(x,t)$ | Amplitude for $t = 10, \ldots$

$\alpha = 0.5, \;\; \omega = 1, \;\; \omega \geq \alpha^2/2,$ $\quad q_0(x) \equiv 0, \; g_0(t) = \alpha e^{2i\omega t} + O(e^{-10t^2})$

Numerics for $-3\alpha^2 < \omega < \alpha^2/2$

Amplitude of $q(x,t)$



$\alpha = 0.5$
$\omega = -2\alpha^2 = -0.5$

$\alpha = 0.5$
$\omega = -\alpha^2 = -0.25$

$q_0(x) \equiv 0, \qquad g_0(t) = \alpha\, e^{2i\omega t} + O(e^{-10t^2})$

Plan

(i) Inverse Scattering Transform (IST) for integrable nonlinear equations on the line

- Lax pair (zero curvature) representation
- Riemann-Hilbert problem
- long time asymptotics for problems on zero background
- long time asymptotics for problems on non-zero background (step-like background)

(ii) Inverse Scattering Transform for integrable nonlinear equations on the half-line

- Riemann-Hilbert problem
- Global Relation
- long time asymptotics for problems with vanishing boundary conditions
- long time asymptotics for problems with non-vanishing boundary conditions

Integrable nonlinear equations

A nonlinear PDE in dimension 1+1 $q_t = F(q, q_x, ...)$ integrable $\Leftrightarrow$ it is compatibility condition for 2 linear equations (Lax pair): matrix-valued (2×2); involve parameter k

- $$\Psi_x = U\Psi, \quad \Psi_t = V\Psi$$
$$U = U(q; k), \quad V = V(q, q_x, \ldots; k)$$

- $q_t = F(q, q_x, ...) \Longleftrightarrow \Psi_{xt} = \Psi_{tx}$ for all k: $\boxed{U_t - V_x = [V, U]}$

Cauchy (whole line) problem: given $q(x, 0) = q_0(x)$, $x \in (-\infty, \infty)$, find $q(x, t)$.

In the case of NLS $\quad \mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0$:

$$U = -\mathrm{i}k\sigma_3 + Q; \quad V = -2\mathrm{i}k^2\sigma_3 + \tilde{Q}$$

with $Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$, $\tilde{Q} = 2kQ + \begin{pmatrix} \mathrm{i}|q|^2 & \mathrm{i}q_x \\ \mathrm{i}\bar{q}_x & -\mathrm{i}|q|^2 \end{pmatrix}$ $\qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Linearized problem, I

The Cauchy problem for linearized NLS – a linear pde

$$\mathrm{i}q_t + q_{xx} = 0, \qquad q(x,0) = q_0(x)$$

is easily solved by Fourier transform:

$$\begin{array}{ccc} q(x,0) & \xrightarrow[\text{Fourier transform}]{\text{direct}} & A(k) \\ {\scriptstyle\text{pde}}\Big\downarrow & & \Big\downarrow{\scriptstyle\text{evolution}} \\ q(x,t) & \xleftarrow[\text{Fourier transform}]{\text{inverse}} & A(k,t) \end{array}$$

The evolution of $A(k) := \hat{q}_0(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} q_0(x) \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$ at time t is given by $A(k,t) = A(k)\mathrm{e}^{-\mathrm{i}\omega(k)t}$ with $\omega(k) = k^2$. Then

$$\boxed{q(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \mathrm{e}^{-\mathrm{i}[kx + k^2 t]}\,\mathrm{d}k}$$

Linearized problem, II

The integral representation for $q(x,t)$ allows studying its long time behavior (via stationary phase/steepest descent method for oscillatory integrals). Let $\xi := \frac{x}{t}$. Then

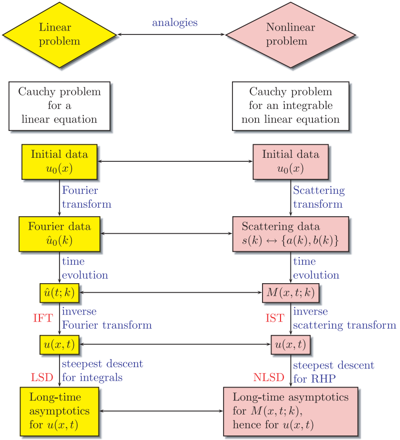
$$q(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \mathrm{e}^{\mathrm{i}t\Phi(\xi;k)} \, \mathrm{d}k$$

with $\Phi(\xi;k) = -\xi k - \omega(k) = -\xi k - k^2$.

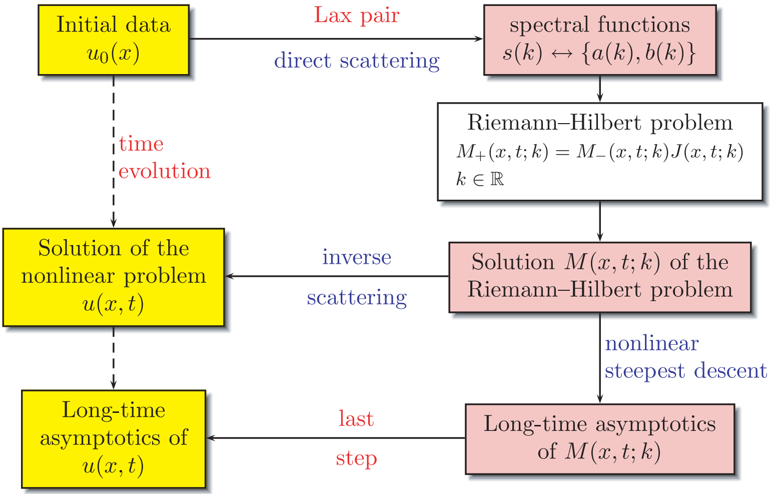
- stationary point: $\Phi'(\xi;k_0) = 0 \Longrightarrow k_0 = -\frac{\xi}{2}$
- $\Phi(\xi;k_0) = \frac{\xi^2}{4}$
- asymptotics:

$$q(x,t) = \frac{1}{\sqrt{t}} \sqrt{\pi} \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} \hat{q}_0 \left(-\frac{\xi}{2}\right) \mathrm{e}^{\frac{\mathrm{i}t\xi^2}{4}} + O(t^{-1})$$

Linear / Nonlinear: IST as nonlinear Fourier transform



Fourier data $\Longleftrightarrow$ Scattering data = "Nonlinear Fourier data"
Integral representation $\Longleftrightarrow$ Riemann-Hilbert (RH) representation
Steepest descent for integrals $\Longleftrightarrow$ Nonlinear steepest descent for RHPs


Initial data
$u_0(x)$
Lax pair
direct scattering
spectral functions
$s(k) \leftrightarrow \{a(k), b(k)\}$
time
evolution
Riemann–Hilbert problem
$M_+(x,t;k) = M_-(x,t;k)J(x,t;k)$
$k \in \mathbb{R}$
Solution of the
nonlinear problem
$u(x,t)$
inverse
scattering
Solution $M(x,t;k)$ of the
Riemann–Hilbert problem
nonlinear
steepest descent
Long-time
asymptotics of
$u(x,t)$
last
step
Long-time asymptotics
of $M(x,t;k)$

Cauchy (whole line) problem: for NLS: given $q(x,0) = q_0(x)$, $x \in (-\infty, \infty)$ ($q_0(x) \to 0$ as $|x| \to \infty$), find $q(x,t)$.

Solution: $q(x,0) \to s(k;0) \to s(k;t) \to q(x,t)$.

- $q(x,0) \to s(k;0)$: direct spectral (scattering) problem for x-equation of the Lax pair
- $s(k;0) \to s(k;t)$: evolution of spectral functions (linear!)
- $s(k;t) \to q(x,t)$: inverse spectral (scattering) problem for x-equation: Riemann–Hilbert problem

In the case of NLS: $U_t - V_x = [V, U]$ with

$U = -\mathrm{i}k\sigma_3 + Q; \quad V = -2\mathrm{i}k^2\sigma_3 + \tilde{Q} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

where $Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$ and $\tilde{Q} = 2kQ + \begin{pmatrix} \mathrm{i}|q|^2 & \mathrm{i}q_x \\ \mathrm{i}\bar{q}_x & -\mathrm{i}|q|^2 \end{pmatrix}$.

- direct scattering: introduce $\Psi_\pm$ dedicated (Jost) solutions of $\Psi_x = U(q(x,t);k)\Psi$:

$$\boxed{\Psi_\pm \sim \Psi_0(= \mathrm{e}^{-\mathrm{i}kx\sigma_3}), \quad x \to \pm\infty}$$

  Then $\Psi_+(x;t,k) = \Psi_-(x;t,k)s(k;t)$ (scattering relation). Particularly, at $t = 0$: $s(k;0) = \Psi_-^{-1}(x;0,k)\Psi_+(x;0,k)$ is determined by $q_0(x)$.

- evolution of scattering functions: using t-equ of Lax pair

$$s_t = 2ik^2[s,\sigma_3] \quad \Rightarrow \quad s(k;t) = \mathrm{e}^{-\mathrm{i}2k^2t\sigma_3}s(k;0)\mathrm{e}^{\mathrm{i}2k^2t\sigma_3}$$

- $s(k;t) \to q(x,t)$: inverse scattering problem for x-equ. Can be done in terms of

Riemann-Hilbert problem (RHP)

Find M: 2×2, piecewise analytic in $\mathbb{C}$ (w.r.t. k) s.t.

- $M_+(x,t;k) = M_-(x,t;k)\mathrm{e}^{-\mathrm{i}(2k^2t+kx)\sigma_3} J_0(k)\mathrm{e}^{\mathrm{i}(2k^2t+kx)\sigma_3}, \ k \in \mathbb{R}$
  $(s(k;0) \to J_0(k)$: algebraic manipulations)
- $M \to I$ as $|k| \to \infty$
- in case x-equ has discrete eigenvalues: $M \equiv (M^{(1)} \ M^{(2)})$ piecewise meromorphic, with residue conditions at poles $\{k_j\}_1^N$

$$Res_{k=k_j} M^{(1)}(x,t,k) = \gamma_j \mathrm{e}^{-2\mathrm{i}(k_j x + 2k_j^2 t)} M^{(2)}(x,t,k_j)$$

Then
$$\boxed{q(x,t) = 2i \lim_{k\to\infty} (kM_{12}(x,t,k))}$$

Hint: M is constructed from columns of Ψ_+ and Ψ_- following their analyticity properties w.r.t k; then jump relation for RHP is a re-written scattering relation for $\Psi_\pm$.

RHP: given contour Σ and $J(k)$, $k \in \Sigma$, find $M(k)$ matrix-valued (2×2):

- $M(k)$ analytic $\mathbb{C} \setminus \Sigma$
- $M_+(k) = M_-(k)J(k)$, $k \in \Sigma$ $\quad M_\pm(k) = \lim_{k' \to k} M(k')$, $k' \in \pm$side of Σ
- $M(k) \to I$ as $k \to \infty$

Reducing to integral equation: solution $M(k)$ is given by $M(k) = I + \big(C(\mu w)\big)(k)$, where:

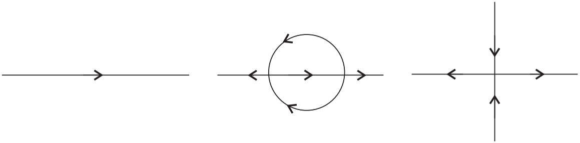
- $w(k) = J(k) - I$
- $(Cf)(k) = \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{f(s)\,\mathrm{d}s}{s-k}$, $k \notin \Sigma$ Cauchy integral
- $\mu(k) \in I + L^2(\Sigma)$: solution of integral equation $(I - C_w)\mu = I$, where:
  - $(C_w f)(k) = C_-(fw)(k)$
  - $(C_\pm f)(k) = \lim_{k' \to k}(Cf)(k')$, $k' \in \pm$side of Σ

Résumé: the Inverse Scattering Transform (IST) method: a kind of nonlinear Fourier transform; change of variables that linearizes the evolution.

Importance: most efficient for studying long-time behavior of solutions of Cauchy problem with general initial data. This is due to explicit (x, t)-dependence of data for the RHP (jump matrix; residue conds. if any), which makes possible to apply a nonlinear version of the steepest descent method (Deift, Zhou, 1993) for studying asymptotic behavior of solutions of relevant Riemann–Hilbert problems with oscillatory jump conditions (linear analogue: asymptotic evaluation of contour integrals by steepest descent or stationary phase methods).

Riemann-Hilbert problems, II

Typical contours Γ:



Jump condition: $M_+(x,t,k) = M_-(x,t,k)J(x,t,k),\ k \in \Sigma$

- x, t: parameters
- jump J: oscillatory behavior w.r.t. t

$$J(x,t,k) = \mathrm{e}^{-\mathrm{i}(2k^2t+kx)\sigma_3} J_0(k) \mathrm{e}^{\mathrm{i}(2k^2t+kx)\sigma_3}$$

- idea of nonlinear steepest descent: deform the contour so that the jumps decay, as $t \to \infty$, to identity matrix or diagonal matrix or constant (w.r.t. k) matrices: the associated RHP can be solved explicitly.

Riemann-Hilbert problems, III

Examples of explicitly solved RHPs (if no res. conds.):

- $J(k) \equiv I \implies M(k) \equiv I$
- $J(k) \equiv diag\{J_1(k)\ J_2(k)\} \implies M(k) \equiv diag\{M_1(k)\ M_2(k)\}$

  with $M_j(k) = \exp\left\{\frac{1}{2\pi i}\int_\Sigma \frac{\log J_j(s)}{s-k}\,\mathrm{d}s\right\}$
- $J(k) \equiv \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $k \in \Sigma = (k_1, k_2)$

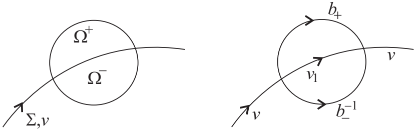
  $\implies M(k) = \frac{1}{2}\begin{pmatrix} \varkappa(k) + \varkappa^{-1}(k) & \varkappa(k) - \varkappa^{-1}(k) \\ \varkappa(k) - \varkappa^{-1}(k) & \varkappa(k) + \varkappa^{-1}(k) \end{pmatrix}$

  with $\varkappa(k) = \left(\frac{k-k_1}{k-k_2}\right)^{1/4}$

  - $J(k) \equiv I$ with nontrivial res. conds.: RHP reduces to linear system of algebraic equations

Riemann-Hilbert problems, IV

Contour deformation:



(a) (b)

Original RHP: $M_+(k) = M_-(k)v(k)$, $k \in \Sigma$.
If $v(k)$ can be factorized as $v(k) = b_-^{-1}(k)v_1(k)b_+(k)$, where $b_\pm(k)$ can be analytically continued into $\Omega^\pm$, then the original RHP is equivalent to:

$$\tilde{M}_+(k) = \tilde{M}_-(k)\tilde{v}(k), \qquad k \in \Sigma \cup \partial\Omega,$$

$$\tilde{M} = \begin{cases} Mb_+^{-1}, & k \in \Omega^+ \\ Mb_-^{-1}, & k \in \Omega^- \\ M, & \text{otherwise} \end{cases}, \qquad \tilde{v} = \begin{cases} v_1, & k \in \Sigma \cap \Omega \\ b_+, & k \in \partial\Omega^+ \setminus \Sigma \\ b_-^{-1}, & k \in \partial\Omega^- \setminus \Sigma \\ v, & k \in \Sigma \setminus \Omega \end{cases}$$

Useful, if $b_\pm = b_\pm(t,k)$ s.t.: oscillating (w.r.t. t) on Σ but decaying, as $t \to \infty$, on $\partial\Omega^\pm \setminus \Sigma$.

RHP for NLS on the line with zero backround, I

Scattering data: $s(k,0) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}(k)$, $k \in \mathbb{R}$

Jump for RHP:

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \qquad k \in \mathbb{R}$$

where

$$\begin{aligned} J(x,t,k) &= \mathrm{e}^{-\mathrm{i}(2k^2 t + kx)\sigma_3} J_0(k) \mathrm{e}^{\mathrm{i}(2k^2 t + kx)\sigma_3} \\ &\equiv \mathrm{e}^{-\mathrm{i}t\theta(\xi,k)\sigma_3} J_0(k) \mathrm{e}^{\mathrm{i}t\theta(\xi,k)\sigma_3} \end{aligned}$$

with $J_0(k) = \begin{pmatrix} 1 + |r(k)|^2 & \bar{r}(k) \\ r(k) & 1 \end{pmatrix}$ $\left(r(k) = \frac{\bar{b}(k)}{a(k)}\right.$ reflection coef$\left.\right)$

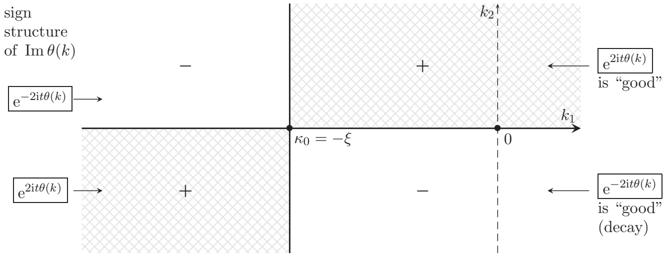
$$\theta(\xi,k) = 4\xi k + 2k^2, \quad \xi = x/4t$$

In accordance with signature table for $\operatorname{Im}\theta$, two algebraic factorizations of jump matrix:

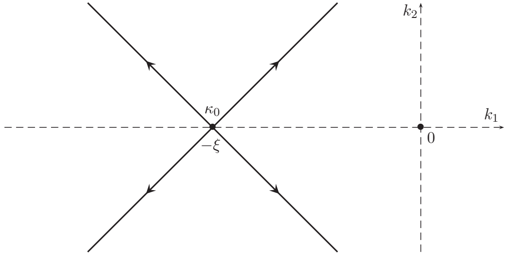
$$\begin{aligned} J &= \begin{pmatrix} 1 & re^{-2i\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{r}e^{2i\theta} & 1 \end{pmatrix} \quad (k > k_0 = -\xi) \\ &= \begin{pmatrix} 1 & 0 \\ \frac{\bar{r}e^{2i\theta}}{1+|r|^2} & 1 \end{pmatrix} \begin{pmatrix} 1 + |r|^2 & 0 \\ 0 & \frac{1}{1+|r|^2} \end{pmatrix} \begin{pmatrix} 1 & \frac{re^{-2i\theta}}{1+|r|^2} \\ 0 & 1 \end{pmatrix} \quad (k < k_0). \end{aligned}$$

RHP for NLS on the line with zero backround, II

Signature table:



Contour deformation:



Asymptotics for NLS on the line with zero background

- As $t \to \infty$, jump matrix decays to I uniformly outside any small vicinity of $k = k_0$
- consequently, the main contribution to the asymptotics comes from the contour in this vicinity (cf. linear case!). After rescaling, the RHP "on small cross" reduces to the RHP on the infinite cross with constant jump, solved in terms of parabolic cylinder functions
- the rescaling leads to amplitude of main term of order $t^{-1/2}$.
- if there are no res. conds.:

$$q(x,t) = \frac{1}{\sqrt{t}}\,\rho(-\xi)\mathrm{e}^{4\mathrm{i}\xi^2 t + 2\mathrm{i}\rho^2(-\xi)\log t + \mathrm{i}\phi(-\xi)} + o\Big(\frac{1}{\sqrt{t}}\Big) \qquad (\xi = \frac{x}{4t})$$

where $\rho^2(k) = \frac{1}{4\pi}\log(1+|r(k)|^2)$,
$\phi(k) = 6\rho^2(k)\log 2 + \frac{3\pi}{4} + arg\ r(k) + arg\ \Gamma(-2i\rho^2(k)) + 4\int_{-\infty}^{k}\log|s-k|d\rho^2(s)$

- in case there are res. cond., the long time behavior is dominated by solitons (breathers): along directions $\xi = \xi_j + O(t^{-1})$, $\xi_j := \mathrm{Re}(k_j)$, $\eta_j := \mathrm{Im}(k_j)$,

$$q(x,t) = -\frac{2\eta_j \exp[-2i\xi_j x - 4i(\xi_j^2 - \eta_j^2)t - i\phi_j]}{\cosh[2\eta_j(x + 4\xi_j t) - \Delta_j]} + O\Big(\frac{1}{t}\Big)$$

NLS on the line with non-zero background, I

Question: Is it possible to develop RHP approach for Cauchy (whole line) problems with initial data that do not decay to 0 as $|x| \to \infty$?

Cauchy problem for focusing NLS with step-like initial data:

- $iq_1 + q_{xx} + 2|q|^2 q = 0$
- $q(x,0) = q_0(x)$
- $q_0(x) \quad \begin{array}{ccc} \to & Ae^{-2iBx}, & x \to +\infty \\ \to & 0, & x \to -\infty \end{array}$

NLS on the line with non-zero background, II

- exact background solution of NLS: $q_0(x,t) = A\mathrm{e}^{-2\mathrm{i}Bx+2i\omega t}$, where $\omega := A^2 - 2B^2$. Then $q(x,t)$ is sought so that $q(x,t) \to q_0(x,t)$ as $x \to +\infty$ and $q(x,t) \to 0$ as $x \to -\infty$ for all t.
- solution of Lax pair associated with $q_0(x,t)$:

$$\Psi_0(x,t,k) = \mathrm{e}^{(-\mathrm{i}Bx+\mathrm{i}\omega t)\sigma_3}\mathcal{E}_0(k)\mathrm{e}^{(-\mathrm{i}X(k)x-\mathrm{i}\Omega(k)t)\sigma_3}$$

where $X(k) = k - B$, $\Omega(k) = 2k^2 + \omega$,

$$\mathcal{E}_0(k) = \frac{1}{2}\begin{pmatrix} \varkappa(k) + \varkappa^{-1}(k) & \varkappa(k) - \varkappa^{-1}(k) \\ \varkappa(k) - \varkappa^{-1}(k) & \varkappa(k) + \varkappa^{-1}(k) \end{pmatrix}$$

with $\varkappa(k) = \left(\frac{k-E_0}{k-\bar{E}_0}\right)^{1/4}$, $E_0 := B + \mathrm{i}A$

- Let $q(x,t) \to q_0(x,t)$ as $x \to \infty$. Then one can define Jost solutions $\Psi_\pm(x,t,k)$ of Lax pair:

$$\begin{aligned} &\Psi_+ \sim \Psi_0, && x \to +\infty, && k \in \Gamma = \mathbb{R} \cup (E_0, \bar{E}_0) \\ &\Psi_- \sim \mathrm{e}^{-ikx\sigma_3}, && x \to -\infty, && k \in \mathbb{R} \end{aligned}$$

- scattering: $\Psi_+ = \Psi_- s(k)$, $k \in \mathbb{R}$; $s(k) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}(k)$

NLS on the line with non-zero background, III

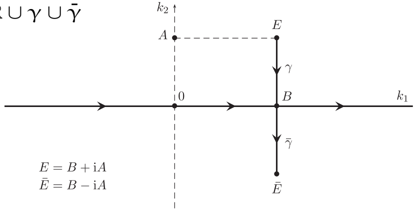
Constructing the RHP:

- $$M(x,t,k) := \begin{cases} \begin{pmatrix} \frac{\Psi_-^{(1)}}{a(k)} & \Psi_+^{(2)} \end{pmatrix} \mathrm{e}^{(\mathrm{i}kx+2\mathrm{i}k^2t)\sigma_3}, & k \in \mathbb{C}_+, \\ \begin{pmatrix} \Psi_+^{(1)} & \frac{\Psi_-^{(2)}}{\bar{a}(\bar{k})} \end{pmatrix} \mathrm{e}^{(\mathrm{i}kx+2\mathrm{i}k^2t)\sigma_3}, & k \in \mathbb{C}_-. \end{cases}$$

NLS on the line with non-zero background, IV

- Jump contour:

$$\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma}$$



- Jump condition for RHP:

$$M_+(x,t;k) = M_-(x,t;k)\mathrm{e}^{-\mathrm{i}(2k^2t+kx)\sigma_3} J_0(k)\mathrm{e}^{\mathrm{i}(2k^2t+kx)\sigma_3}, \quad k \in \Sigma = \mathbb{R} \cup (E_0, \bar{E}_0)$$

$$\text{where } J_0(k) = \begin{cases} \begin{pmatrix} 1+|r(k)|^2 & \bar{r}(k) \\ r(k) & 1 \end{pmatrix}, & k \in \mathbb{R} \\ \begin{pmatrix} h(k) & i \\ 0 & h^{-1}(k) \end{pmatrix}, & k \in (E_0, \bar{E}_0) \cap \mathbb{C}_+ , \\ \begin{pmatrix} \bar{h}^{-1}(\bar{k}) & 0 \\ i & \bar{h}(\bar{k}) \end{pmatrix}, & k \in (E_0, \bar{E}_0) \cap \mathbb{C}_- \end{cases}$$

$h(k) = \frac{a_-(k)}{a_+(k)}$, $r(k) = \frac{\bar{b}(k)}{a(k)}$

NLS on the line with non-zero background, V

More general background: finite band (finite gap) potential

- $\Psi_0(x,t,k) = \mathrm{e}^{(\mathrm{i}f_0 x + \mathrm{i}g_0 t)\sigma_3}\mathcal{E}_N(x,t,k)\mathrm{e}^{-(\mathrm{i}f(k)x+\mathrm{i}g(k)t)\sigma_3}$

  (i) $f(k)$, $g(k)$: solve scalar RHPs (can be written in terms of Cauchy integrals):

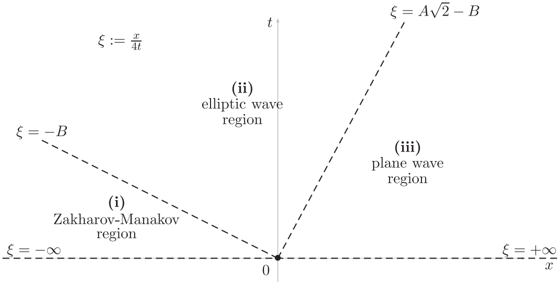
  - $f_+(k) + f_-(k) = C_j^f$, $k \in \Gamma_j = (E_j, \bar{E}_j)$, $\qquad j = 0, \dots, N$;
    $f(k) = k + f_0 + O(k^{-1})$, $k \to \infty$;
  - $g_+(k) + g_-(k) = C_j^g$, $k \in \Gamma_j = (E_j, \bar{E}_j)$; $g(k) = 2k^2 + g_0 + O(k^{-1})$, $k \to \infty$
  - $C_0^f = C_0^g = 0$; $\{C_j^f, C_j^g\}_1^N$, f_0, g_0: all uniquely determined by $\{E_j\}_0^N$

  (ii) $\mathcal{E}_N(x,t,k)$: solves matrix RHP with piecewise constant jump:

$$\mathcal{E}_{N+}(k) = \mathcal{E}_{N-}(k)J_j, \quad J_j = \begin{pmatrix} 0 & \mathrm{i}\mathrm{e}^{-\mathrm{i}(C_j^f x + C_j^g t + \phi_j)} \\ \mathrm{i}\mathrm{e}^{\mathrm{i}(C_j^f x + C_j^g t + \phi_j)} & 0 \end{pmatrix}, \; k \in (E_j, \bar{E}_j)$$

- $\mathcal{E}_N(x,t,k)$ (and, consequently, $q_N(x,t)$) can be explicitly written in terms of multidimensional Riemann theta functions and Abelian integrals

In the case $q_0(x) \to A\mathrm{e}^{-2\mathrm{i}Bx}$ as $x \to +\infty$, $q_0(x) \to 0$ $x \to -\infty$:



Three sectors in the (x, t) half-plane, where $q(x, t)$ behaves differently for large t, depending on the magnitude of $\xi = x/4t$.

(i) $\xi < -B$: slowly decaying ($t^{-1/2}$) self-similar wave, as in the case of zero background

$$q(x,t) = \frac{1}{\sqrt{t}}\,\rho(-\xi)\mathrm{e}^{4\mathrm{i}\xi^2 t + 2\mathrm{i}\rho^2(-\xi)\log t + \mathrm{i}\phi(-\xi)} + O(t^{-1})$$

(ii) $-B < \xi < -B + A\sqrt{2}$: oscillations governed by modulated elliptic wave

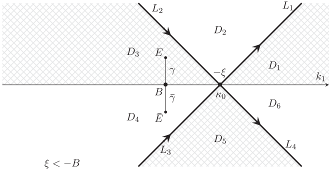
(iii) $\xi > -B + A\sqrt{2}$: plane wave

$$q(x,t) = A\mathrm{e}^{2\mathrm{i}(\omega t - Bx - \phi(\xi))} + O(t^{-1/2})$$

- Asymptotic formulas follow from explicit solutions of model (limiting) RHP obtained after transformations

Model problems:

(i) $\xi < -B$:



(ii) $-B < \xi < -B + A\sqrt{2}$: two arcs: $(E_0, \bar{E}_0)$ and $(\alpha(\xi), \bar{\alpha}(\xi))$.

$$J_{mod} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \; k \in (E_0, \bar{E}_0);$$

$$J_{mod} = \begin{pmatrix} 0 & \mathrm{i}e^{-\mathrm{i}(C^f(\xi)x + C^g(\xi)t + \phi(\xi))} \\ \mathrm{i}e^{\mathrm{i}(C^f(\xi)x + C^g(\xi)t + \phi(\xi))} & 0 \end{pmatrix}, \; k \in (\alpha(\xi), \bar{\alpha}(\xi)).$$

(iii) $\xi > -B + A\sqrt{2}$: one arc $(E_0, \bar{E}_0)$; $J_{mod} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$

- $B < \xi < -B + A\sqrt{2}$: modulated elliptic wave

$$q(x,t) = \hat{A}\frac{\theta_3(\beta t + \gamma)}{\theta_3(\beta t + \tilde{\gamma})}\mathrm{e}^{2\mathrm{i}(\nu t - \phi)} + O(t^{-1/2}).$$

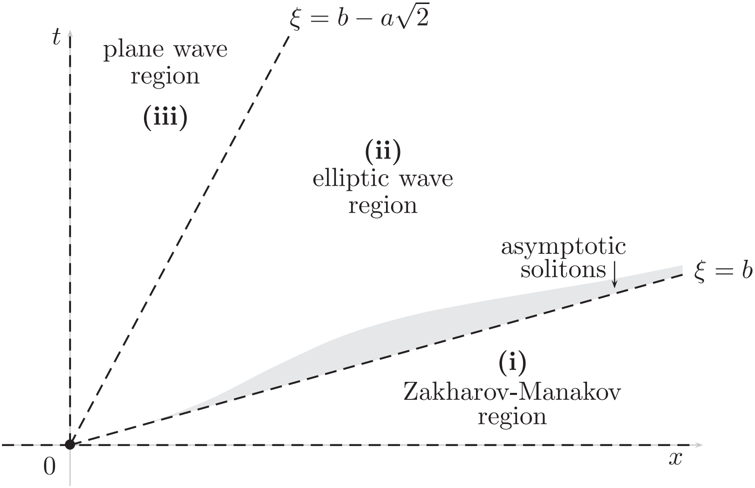
  Here $\hat{A}$, β, γ, $\tilde{\gamma}$, ν, ϕ are functions of ξ.

- $\theta_3(z) = \sum_{m\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau m^2 + 2\pi\mathrm{i}mz}$ is a theta function of invariant $\tau(\xi)$

Two observations concerning solutions of the whole line (Cauchy) problems restricted to half-line $x > 0$:

(i) In the step-like problem: change of variable $x \mapsto -x$ leads to problem with ini. data $q_0(x) \to 0$ as $x \to +\infty$ and $q_0(x) \to A\mathrm{e}^{2\mathrm{i}Bx}$ as $x \to -\infty$. Then lines $\xi = B$ and $\xi = B - A\sqrt{2}$ separate sectors with different behavior of $q(x,t)$.

If $B - A\sqrt{2} > 0$, then the t-axis ($x = 0$) lies in sector with plane wave behavior: $q(x,t) \sim A\mathrm{e}^{2\mathrm{i}Bx+2\mathrm{i}\omega t}$. Particularly, at $x = 0$: $q(0,t) \sim A\mathrm{e}^{2\mathrm{i}\omega t}$! Thus in this case we have examples of solutions of NLS equation which, being considered in $x > 0$, $t > 0$, exhibit decaying (as $x \to +\infty$) ini. values (at $t = 0$) and (asymptotically) periodic boundary values at the boundary $x = 0$.


$\xi = b - a\sqrt{2}$
t
plane wave region
(iii)
(ii)
elliptic wave region
asymptotic solitons
$\xi = b$
(i)
Zakharov-Manakov region
0
x

(ii) In the problem on zero background (with decaying ini. data), let ini. data be such that spectral function $a(k)$ has a single zero located on the imaginary axis: $a(k_0) = 0$ with $k_0 = \mathrm{i}\sqrt{\frac{\omega}{2}}$, $\omega > 0$. Then:

- if $r(k) \equiv 0$, $k \in \mathbb{R}$, then the RHP with single res. cond.

$$Res_{k=\mathrm{i}\nu} M^{(1)}(x,t,k) = \gamma_0 \mathrm{e}^{2\sqrt{2\omega}x} \mathrm{e}^{2\mathrm{i}\omega t} M^{(2)}(x,t,\mathrm{i}\nu)$$

  and trivial jump cond. $(J \equiv I)$ reduces to a system of linear algebraic equations; solving this leads to

$$q(x,t) = \frac{\sqrt{2\omega}}{\cosh(\sqrt{2\omega}x + \phi_0)} \mathrm{e}^{2\mathrm{i}\omega t}$$

  which is exact solution (stationary soliton) of NLS s.t.
  - $q(x,0) \to 0$ as $x \to \infty$
  - $q(0,t) = A\mathrm{e}^{2\mathrm{i}\omega t}$ with $A = \frac{\sqrt{2\omega}}{\cosh(\phi_0)}$
- for general $r(k)$, $q(x,t)$ approaches, as $t \to \infty$, the stationary soliton.

General scheme for boundary value problems via IST

Natural problem: to adapt (generalize) the RHP approach to boundary-value (initial-boundary value) problems for integrable equations.

Data for an IBV problem (e.g, in domain $x > 0$, $t > 0$):

(i) Initial data: $q(x,0) = q_0(x)$, $x > 0$

(ii) Boundary data: $q(0,t) = g_0(t)$, $q_x(0,t) = g_1(t), \dots$.

Question: How many boundary values?

For a well-posed problem: roughly "half" the number of x-derivatives.

For NLS: one b.c. (e.g., $q(0,t) = g_0(t)$).

General idea for IBV: use both equations of the Lax pair as spectral problems.

Common difficulty: spectral analysis of the t-equation on the boundary ($x = 0$) involves more functions (boundary values $q(0,t), q_x(0,t), \dots$) than possible data for a well-posed problem.

Half-line problem for NLS

For NLS: t-equation involves q and q_x; hence for the (direct) spectral analysis at $x = 0$ one needs $q(0,t)$ and $q_x(0,t)$. Assume that we are given the both. Then one can define two sets of spectral functions coming from the spectral analysis of x-equation and t-equation.

(i) $q_0 \mapsto \{a(k), b(k)\}$ (direct problem for x-equ); $\quad s \equiv \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$

$\{g_0, g_1\} \mapsto \{A(k), B(k)\}$ (direct problem for t-equ)

(ii) From the spectral functions $\{a(k), b(k), A(k), B(k)\}$, the jump matrix $J(x,t,k)$ for the Riemann-Hilbert problem is constructed: $\{a(k), b(k), A(k), B(k)\} \mapsto J_0(k)$:

$$J(x,t,k) = \mathrm{e}^{-\mathrm{i}(2k^2 t + kx)\sigma_3} J_0(k) \mathrm{e}^{\mathrm{i}(2k^2 t + kx)\sigma_3}$$

(notice the same explicit dependence on (x,t)!) The jump conditions are across a contour Σ determined by the asymptotic behavior of $g_0(t)$ and $g_1(t)$

(iii) The RHP is formulated relative to Σ:
$M_+(x,t,k) = M_-(x,t,k)J(x,t,k),\ k \in \Sigma; \qquad M \to I$ as $k \to \infty$

(iv) Similarly to the Cauchy (whole-line) problem, the solution of the IBV (half-line) problem is given in terms of the solution of the RHP: $q(x,t) = 2\mathrm{i} \lim_{k\to\infty} (kM_{12}(x,t,k))$

Given $q(x,t)$, how to construct $M(x,t,k)$?

Define $\Psi_j(x,t,k)$, $j = 1,2,3$ solutions (2×2) of the Lax pair equations normalized at "corners" of the (x,t)-domain where the IBV problem is formulated:

1. $\Psi_1(0,T,k) = \mathrm{e}^{-2\mathrm{i}k^2 T\sigma_3}$ $(\Psi_1(0,t,k) \simeq \mathrm{e}^{-2\mathrm{i}k^2 t\sigma_3}$ as $t \to \infty)$
2. $\Psi_2(0,0,k) = I$
3. $\Psi_3(x,0,k) \simeq \mathrm{e}^{-\mathrm{i}kx\sigma_3}$ as $x \to \infty$

Being simultaneous solutions of x-and t-equation, they are related by two scattering relations:

(i) $\Psi_3(x,t,k) = \Psi_2(x,t,k)s(k)$ $\qquad s = \begin{pmatrix} \bar{a} & b \\ -\bar{b} & a \end{pmatrix}$

(ii) $\Psi_1(x,t,k) = \Psi_2(x,t,k)S(k;T)$ $\qquad S = \begin{pmatrix} \bar{A} & B \\ -\bar{B} & A \end{pmatrix}$

Then M is constructed from columns of Ψ_1, Ψ_2 and Ψ_3 following their analyticity and boundedness properties w.r.t k, and the jump relation for RHP is re-written scattering relations (i)+(ii) for Ψ_j.

For NLS in half-strip $(T < \infty)$ or in quarter plane $(T = \infty)$ with $g_j(t) \to 0$ as $t \to \infty$: first column of $\Psi_1(x,t,k)\mathrm{e}^{(-\mathrm{i}kx - 2\mathrm{i}k^2 t)\sigma_3}$ is bounded in $\{k : \operatorname{Im} k \geq 0, \operatorname{Im} k^2 \leq 0\}$, etc., which leads to $\Sigma = \mathbb{R} \cup \mathrm{i}\mathbb{R}$.

Direct spectral problems for NLS in half-strip $x > 0,\ 0 < t < T$

- Given $q_0(x)$, determine $a(k)$, $b(k)$: $\boxed{a(k) = \Phi_2(0,k), \quad b(k) = \Phi_1(0,k)}$, where vector $\Phi(x,k)$ is the solution of the x-equation evaluated at $t = 0$:

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = Q(x, {\color{red}0}, k)\Phi, \quad 0 < x < \infty, \mathrm{Im}\, k \geq 0$$

$$\Phi(x,k) = e^{ikx}\left(\begin{pmatrix}0\\1\end{pmatrix} + o(1)\right) \text{ as } x \to \infty,$$

$$Q(x,0,k) = \begin{pmatrix} 0 & q_0(x) \\ -\bar{q}_0(x) & 0 \end{pmatrix}$$

- Given $\{g_0(t), {\color{red}g_1(t)}\}$, determine $A(k;T), B(k;T)$: $\boxed{A(k;T) = \mathrm{e}^{2\mathrm{i}k^2T}\overline{\tilde{\Phi}_1(T,\bar{k})}, \quad B(k;T) = -\mathrm{e}^{2\mathrm{i}k^2T}\tilde{\Phi}_2(T,k)}$, where vector $\tilde{\Phi}(x,k)$ is the solution of the t-equation evaluated at $x = 0$:

$$\tilde{\Phi}_t + 2\mathrm{i}k^2\sigma_3\tilde{\Phi} = \tilde{Q}({\color{red}0}, t, k)\tilde{\Phi}, \quad 0 < t < T,$$

$$\tilde{\Phi}(0,k) = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\tilde{Q}(0,t,k) = \begin{pmatrix} -|g_0(t)|^2 & 2kg_0(t) - \mathrm{i}{\color{red}g_1(t)} \\ 2k\bar{g}_0(t) + \mathrm{i}{\color{red}\bar{g}_1(t)} & |g_0(t)|^2 \end{pmatrix}$$

- Contour: $\Sigma = \mathbb{R} \cup \mathrm{i}\mathbb{R}$
- Jump matrix:

$$J_0(k) = \begin{cases} \begin{pmatrix} 1+|r(k)|^2 & \bar{r}(k) \\ r(k) & 1 \end{pmatrix}, & k > 0, \\ \begin{pmatrix} 1 & 0 \\ C(k;T) & 1 \end{pmatrix}, & k \in \mathrm{i}\mathbb{R}_+, \\ \begin{pmatrix} 1 & \bar{C}(\bar{k};T) \\ 0 & 1 \end{pmatrix}, & k \in \mathrm{i}\mathbb{R}_-, \\ \begin{pmatrix} 1+|r(k)+C(k;T)|^2 & \bar{r}(k)+\bar{C}(k;T) \\ r(k)+C(k;T) & 1 \end{pmatrix}, & k < 0, \end{cases}$$

where $r(k) = \dfrac{\bar{b}(k)}{a(k)}$, $C(k;T) = -\dfrac{\overline{B(\bar{k};T)}}{a(k)d(k;T)}$ with $d = a\bar{A} + b\bar{B}$

(also works for $T = +\infty$ if $g_0(t), g_1(t) \to 0,\ t \to \infty$)

Compatibility of boundary values: Global Relation

- The fact that the set of initial and boundary values $\{q_0(x), g_0(t), g_1(t)\}$ cannot be prescribed arbitrarily (as data for IBVP) must be reflected in spectral terms.

  Indeed, from scattering relations (i)+(ii):
  $S^{-1}(k;T)s(k) = \Psi^{-1}(x,t,k)\Psi_3(x,t,k)$. Evaluating this at $x=0$, $t=T$ and using analyticity and boundedness properties of Ψ_j, one deduces for the (12) entry of $S^{-1}s$:

$$\boxed{A(k;T)b(k) - a(k)B(k;T) = O\left(\frac{\mathrm{e}^{4\mathrm{i}k^2 T}}{k}\right), \ k \to \infty}$$

$$k \in D = \{\mathrm{Im}\, k \geq 0, \mathrm{Re}\, k \geq 0\}$$

- This relation is called Global Relation (GR): it characterizes the compatibility of $\{q_0(x), g_0(t), g_1(t)\}$ in spectral terms.

Typical theorem: Consider the IBVP with given $q_0(x)$ and $g_0(t)$. Assume that there exists $g_1(t)$ such that the associated spectral functions $\{a(k), b(k), A(k), B(k)\}$ satisfy the Global Relation. Then the solution of the IBVP is given in terms of the solution of the RHP above. Moreover, it satisfies also the b.c. $q_x(0,t) = g_1(t)$.

IBVP for linearized NLS

- $\mathrm{i}q_t + q_{xx} = 0$
- $q(x,0) = q_0(x)$
- $q(0,t) = g_0(t)$ $\quad$ ($q_x(0,t) = g_1(t)$ is not prescribed for well-posed problem)

Lax Pair:

- $\mu_x + \mathrm{i}k\mu = q$
- $\mu_t + \mathrm{i}k^2\mu = \mathrm{i}q_x + kq$

Another form: $\left(\mu \mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^2t}\right)_x = q\mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^2t}$, $\left(\mu \mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^2t}\right)_t = (\mathrm{i}q_x + kq)\mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^2t}$;
suggests defining 1-form $W := \left(q\mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^2t}\right)\mathrm{d}x + \left((\mathrm{i}q_x + kq)\mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^2t}\right)\mathrm{d}t$
s.t. $W = \mathrm{d}\left(\mu \mathrm{e}^{\mathrm{i}kx+\mathrm{i}k^2t}\right)$.

$$0 = \int_{\square} W,\ X \to \infty:$$

$$\int_0^\infty \mathrm{e}^{\mathrm{i}kx} q_0(x)\,\mathrm{d}x - \mathrm{i}\int_0^t \mathrm{e}^{\mathrm{i}k^2\tau} g_1(\tau)\,\mathrm{d}\tau - k\int_0^t \mathrm{e}^{\mathrm{i}k^2\tau} g_0(\tau)\,\mathrm{d}\tau = \mathrm{e}^{\mathrm{i}k^2t}\int_0^\infty \mathrm{e}^{\mathrm{i}kx} q(x,t)\,\mathrm{d}x$$

valid for $\mathrm{Im}\,k \geq 0$. View this as Global Relation:

$$\hat{q}_0(k) - \mathrm{i}h_1(k,t) - kh_0(k,t) = O\left(\frac{\mathrm{e}^{\mathrm{i}k^2t}}{k}\right), \quad \mathrm{Im}\,k \geq 0, \mathrm{Re}\,k \geq 0$$

Here $\hat{q}_0(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}kx} q_0(x)\,\mathrm{d}x$, $h_j(k,t) = \int_0^t \mathrm{e}^{\mathrm{i}k^2\tau} g_j(\tau)\,\mathrm{d}\tau$, $j = 0, 1$.

Using Global Relation (GR) in linear case: 2 ways, I

1. construct the Dirichlet-to-Neumann map, i.e., derive $g_1(t) = q_x(0,t)$ from $\{q_0(x) = q(x,0),\, g_0(t) = q(0,t)\}$

   Multiply GR by $-\frac{\mathrm{i}k}{\pi}\mathrm{e}^{-\mathrm{i}k^2 t'}$, integrate along ∂D, then $t' \to t$:

$$
\begin{aligned}
g_1(t) = &-\frac{\mathrm{i}}{\pi}\int_{\partial D} \mathrm{d}k \mathrm{e}^{-\mathrm{i}k^2 t} k \left(\int_0^\infty q_0(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x\right) \\
&+ \frac{1}{\pi}\int_{\partial D} \mathrm{d}k \left\{\mathrm{i}k^2 \int_0^t \mathrm{e}^{\mathrm{i}k^2(\tau - t)} g_0(\tau)\,\mathrm{d}\tau - g_0(t)\right\}
\end{aligned}
$$

Using Global Relation (GR) in linear case: 2 ways, II

2. solve the IBVP. (i) from GR, by inverse Fourier:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}\hat{q}_0(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\mathrm{e}^{-\mathrm{i}kx}k\left(\int_0^t \mathrm{e}^{\mathrm{i}k^2(\tau-t)}g_0(\tau)\,\mathrm{d}\tau\right)$$
$$- \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}h_1(k,t) \qquad \left(h_1(k,t) = \int_0^t \mathrm{e}^{\mathrm{i}k^2\tau}g_1(\tau)\,\mathrm{d}\tau\right)$$

(ii) using GR for $-k$ and the symmetry $h_j(-k,t) = h_j(k,t)$:

$$-\mathrm{i}h_1(k,t) = -kh_0(k,t) - \hat{q}_0(-k) + \mathrm{e}^{\mathrm{i}k^2t}\hat{q}(-k,t)$$

(iii) By Jordan's lemma, $\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\hat{q}(-k,t)\,\mathrm{d}k = 0$.

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}(\hat{q}_0(k) - \hat{q}_0(-k))\,\mathrm{d}k$$
$$- \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}k\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}k^2t}k\left(\int_0^t \mathrm{e}^{\mathrm{i}k^2\tau}g_0(\tau)\right)$$

Using Global Relation for the NLS, I

(1) GR can be used to describe the Dirichlet-to-Neumann map:

$$g_1(t) = \frac{g_0(t)}{\pi}\int \mathrm{e}^{-2\mathrm{i}k^2 t}\left(\tilde{\Phi}_2(t,k) - \tilde{\Phi}_2(t,-k)\right) dk + \frac{4\mathrm{i}}{\pi}\int \mathrm{e}^{-2\mathrm{i}k^2 t} k r(k)\overline{\tilde{\Phi}_2(t,\bar{k})} dk$$
$$+ \frac{2\mathrm{i}}{\pi}\int \mathrm{e}^{-2\mathrm{i}k^2 t}(k[\tilde{\Phi}_1(t,k) - \tilde{\Phi}_1(t,-k)] + \mathrm{i}g_0(t))dk \quad \left(\int = \int_{\partial D}\right)$$

But: nonlinear! (g_1 is involved in the construction of $\tilde{\Phi}_j$)

- In the small-amplitude limit, this reduces to a formula giving $g_1(t)$ in terms of $g_0(t)$ and $q_0(x)$ (via $r(k)$); here NLS reduces to a linear equation $\mathrm{i}q_t + q_{xx} = 0$.
- This suggests perturbative approach: given $g_0(t)$ say periodic with small amplitude, derive a perturbation series for $g_1(t)$, with periodic terms.

Using Global Relation for the NLS, II

(2) For some particular b.c. (called linearizable): use additional symmetry ($k \mapsto -k$) in t-equation for expressing all ingredients of jump matrix in terms of spectral data associated with initial data only. Examples: IBVP with homogeneous Dirichlet b.c. ($q(0,t) \equiv 0$); also Neumann b.c. ($q_x(0,t) \equiv 0$) and mixed (Robin) b.c. $q_x(0,t) + \rho q(0,t) \equiv 0$

(i) additional symmetry: $A(-k) = A(k)$, $B(-k) = -\frac{2k+\mathrm{i}\rho}{2k-\mathrm{i}\rho} B(k)$

(ii) global relation: suggests replacing $B(k)/A(k)$ by $b(k)/a(k)$ for $\mathrm{Im}\, k \geq 0, \mathrm{Re}\, k \geq 0$

(i)+(ii) allows "solving" $A(k)$, $B(k)$ in terms of $a(k)$, $b(k)$, so that the jump matrix for RHP can be expressed in terms of $a(k)$ and $b(k)$ (and ρ) only: $C(k)$, $k \in \Sigma$ can be replaced by

$$\tilde{C}(k) = \frac{\bar{b}(-\bar{k})}{a(k)} \frac{2k + \mathrm{i}\rho}{(2k - \mathrm{i}\rho)a(k)\bar{a}(-\bar{k}) - (2k + \mathrm{i}\rho)b(k)\bar{b}(-\bar{k})}$$

Using Global Relation for the NLS, III

(3) For $T = \infty$: if $g_0(t) \to 0$ as $t \to \infty$ and assuming that $g_1(t) \to 0$, the GR takes the form

$$A(k)b(k) - a(k)B(k) = 0, \;\; k \to \infty, \;\; \mathrm{Im}\, k \geq 0, \mathrm{Re}\, k \geq 0$$

Since the structure of the RHP is similar to that for whole-line problem, one can study long-time behavior of solution via nonlinear steepest descent.
But: parameters of the asymptotics - in terms of $A(k), B(k)$, which are not known for a well-posed IBVP.

IBV problem with oscillatory b.c.

For $T = \infty$: the approach can be implemented for boundary values non-decaying as $t \to \infty$. But for this: one needs correct large-time behavior of $g_1(t)$ associated with that of the given $g_0(t)$; this is because both $g_0(t)$ and $g_1(t)$ determine the spectral problem for t-equation and thus the structure of associated spectral functions $A(k)$, $B(k)$.

Dirichlet-to-Neumann map

Let $q(0,t) = \alpha\, \mathrm{e}^{2\mathrm{i}\omega t}$ $(q(0,t) - \alpha\, \mathrm{e}^{2\mathrm{i}\omega t} \to 0,\ t \to \infty)$
Neumann values $(q_x(0,t))$:

(i) numerics:

$$q_x(0,t) \simeq c\, \mathrm{e}^{2\mathrm{i}\omega t} \qquad c = \begin{cases} 2\mathrm{i}\alpha\sqrt{\frac{\alpha^2-\omega}{2}}, & \omega \le -3\alpha^2 \\ \alpha\sqrt{2\omega - \alpha^2}, & \omega \ge \frac{\alpha^2}{2} \end{cases}$$

(ii) theoretical results: agreed with numerics (for all $x > 0$, $t > 0$) provided c as above.

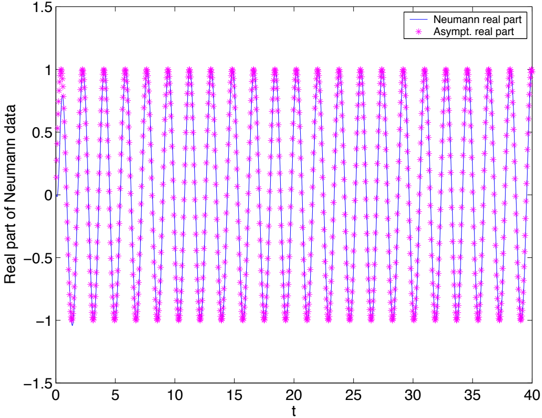
Question: Why these particular values of c?

(the spectral mapping $\{g_0, g_1\} \mapsto \{A(k), B(k)\}$ is well-defined for any $c \in \mathbb{C}$!)

Idea: Use the global relation (its impact on analytic properties of $A(k)$, $B(k)$) to specify admissible values of parameters α, ω, c.

Numerics: Neumann values, $\omega < -3\alpha^2$

Neumann values $q_x(0, t)$ for $\alpha = 0.5$ and $\omega = -1.75$.



The numerics agree with $q_x(0, t) = 2\mathrm{i}\alpha\beta\, q(0, t)$.

Theorem 1: $\omega < -3\alpha^2$

Consider the Dirichlet initial-boundary value problem for NLS_+

- $\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \quad x, t \in \mathbb{R}_+,$
- $q(x,0) = q_0(x)$ fast decaying,
- $q(0,t) = g_0(t) \equiv \alpha\, \mathrm{e}^{2\mathrm{i}\omega t}$ time-periodic, $\alpha > 0$, $\boxed{\omega < -3\alpha^2}$
- $q_0(0) = g_0(0)$.

▷ Assume $q_x(0,t) \sim 2\mathrm{i}\alpha\beta\, \mathrm{e}^{2\mathrm{i}\omega t}$ as $t \to +\infty$ with $\beta = \sqrt{\frac{\alpha^2-\omega}{2}}$.

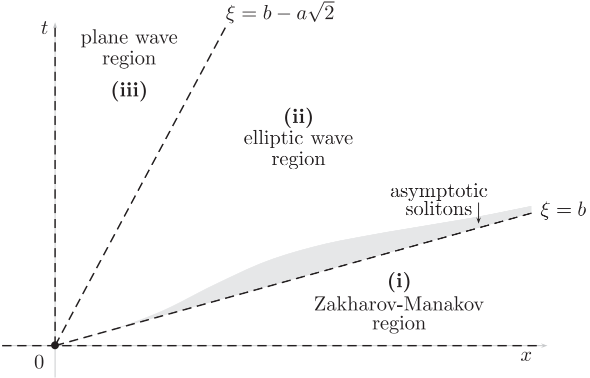
Let $\xi := \frac{x}{4t}$. Then for large t, the solution $q(x,t)$ behaves differently in 3 sectors of the (x,t)-quarter plane:

(i) $\xi > \beta \implies q(x,t)$ looks like decaying modulated oscillations of Zakharov-Manakov type.

(ii) $\sqrt{\beta^2 - 2\alpha^2} < \xi < \beta \implies q(x,t)$ looks like a modulated elliptic wave.

(iii) $0 \le \xi < \sqrt{\beta^2 - 2\alpha^2} \implies q(x,t)$ looks like a plane wave.

Three regions for $\omega < -3\alpha^2$



Regions in the (x, t)-quarter-plane: $\xi = \frac{x}{4t}, \ \beta = \sqrt{\frac{\alpha^2 - \omega}{2}}$

Asymptotics for $\omega < -3\alpha^2$

- $\xi = \frac{x}{4t} > \beta$:

$$q(x,t) = \frac{1}{\sqrt{t}}\,\rho(-\xi)\mathrm{e}^{4\mathrm{i}\xi^2 t + 2\mathrm{i}\rho^2(-\xi)\log t + \mathrm{i}\phi(-\xi)} + \mathrm{o}\Big(\frac{1}{\sqrt{t}}\Big)$$

- $\beta - \alpha\sqrt{2} < \xi < \beta$:

$$q(x,t) \simeq [\alpha + \mathrm{Im}\, d(\xi)] \frac{\theta_3[B_g t/2\pi + B_\omega \Delta/2\pi + U_-]}{\theta_3[B_g t/2\pi + B_\omega \Delta/2\pi + U_+]} \frac{\theta_3[U_+]}{\theta_3[U_-]} \mathrm{e}^{2\mathrm{i}g_\infty(\xi)t - 2\mathrm{i}\phi(\xi)}$$

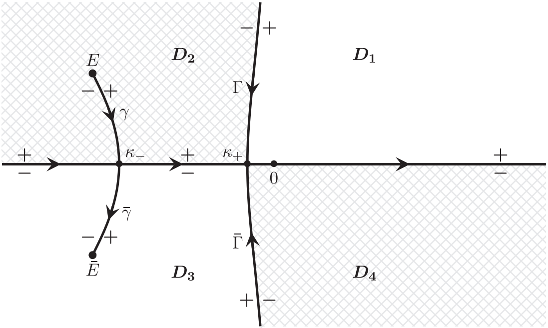
- $0 < \xi < \beta - \alpha\sqrt{2}$:

$$q(x,t) = \alpha\,\mathrm{e}^{2\mathrm{i}[\beta x + \omega t - \phi(\xi)]} + \mathrm{O}\Big(\frac{1}{\sqrt{t}}\Big)$$

The parameters (functions of ξ) $d, B_g, B_\omega, g_\infty, \phi$ can be expressed in terms of the spectral functions associated to IB data $\{q_0(x), \alpha, \omega\}$.

The RHP for NLS: the contour

for $\omega < -3\alpha^2$, assuming $q_x(0,t) \sim 2\mathrm{i}\alpha\beta\, \mathrm{e}^{2\mathrm{i}\omega t}$



$\Sigma = \mathbb{R} \cup \gamma \cup \bar{\gamma} \cup \Gamma \cup \bar{\Gamma}$ with $E = -\beta + \mathrm{i}\alpha$.

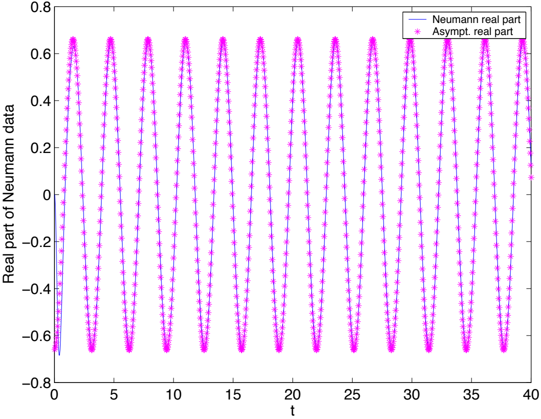
The RHP for NLS: the jump matrix

$$J(x,t;k) = \begin{cases} \begin{pmatrix} 1 & -\bar{\rho}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -\rho(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1+|\rho(k)|^2 \end{pmatrix} & k \in (-\infty, \kappa_+), \\ \begin{pmatrix} 1 & -\bar{r}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ -r(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1+|r(k)|^2 \end{pmatrix} & k \in (\kappa_+, \infty), \\ \begin{pmatrix} 1 & 0 \\ c(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix} & k \in \Gamma, \\ \begin{pmatrix} 1 & \bar{c}(\bar{k})\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \bar{\Gamma}, \\ \begin{pmatrix} 1 & 0 \\ f(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix} & k \in \gamma, \\ \begin{pmatrix} 1 & -\bar{f}(\bar{k})\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \bar{\gamma}. \end{cases}$$

where $\boxed{\theta(k) = \theta(k,\xi) = 2k^2 + 4\xi k}$ with $\boxed{\xi = \frac{x}{4t}}$

Numerics: Neumann values, $\omega \geq \alpha^2/2$

Neumann values $q_x(0,t)$ for $\alpha = 0.5$ and $\omega = 1$.



The numerics agree with $q_x(0,t) = 2\alpha\hat{\beta}\, q(0,t)$.

Theorem 2: $\omega \geq \alpha^2/2$

Consider the Dirichlet initial-boundary value problem for NLS_+

- $\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0, \quad x, t \in \mathbb{R}_+$.
- $q(x,0) = q_0(x)$ fast decaying.
- $q(0,t) = g_0(t) \equiv \alpha\, \mathrm{e}^{2\mathrm{i}\omega t}$ time-periodic, $\alpha > 0$, $\boxed{\omega \geq \alpha^2/2}$
- $q_0(0) = g_0(0)$.

▷ Assume that $q_x(0,t) \sim 2\alpha\hat{\beta}\, \mathrm{e}^{2\mathrm{i}\omega t}$ with $\hat{\beta} = \pm\frac{1}{2}\sqrt{2\omega - \alpha^2}$.

Then for $\xi = \frac{x}{4t} > \varepsilon > 0$,

$$q(x,t) = \frac{1}{\sqrt{t}}\,\rho(-\xi)\mathrm{e}^{4\mathrm{i}\xi^2 t + 2\mathrm{i}\rho^2(-\xi)\log t + \mathrm{i}\phi(-\xi)} + \mathrm{o}\Big(\frac{1}{\sqrt{t}}\Big)$$

(decaying modulated oscillations of Zakharov-Manakov type), where parameters $\rho(\xi)$ and $\phi(\xi)$ are determined by the IB data $q_0(x)$, $g_0(t)$, and $g_1(t)$ via the spectral functions $a(k), b(k), A(k), B(k)$.

Theorem 3: admissible $\{\alpha, \omega, c\}$

Let $q(x,t)$ be a solution of the NLS $(x > 0, t > 0)$ such that:

- $q(0,t) - \alpha\, \mathrm{e}^{2\mathrm{i}\omega t} \to 0$ as $t \to +\infty$ $(\alpha > 0, \omega \in \mathbb{R})$
- $q_x(0,t) - c\, \mathrm{e}^{2\mathrm{i}\omega t} \to 0$ as $t \to +\infty$, for some $c \in \mathbb{C}$
- $q(x,t) \to 0$ as $x \to +\infty$ $(\forall t \geq 0)$

Then the admissible values of $\{\alpha, \omega, c\}$ are given by:

- $\omega \leq -3\alpha^2$, $c = 2\mathrm{i}\alpha\sqrt{\frac{\alpha^2-\omega}{2}}$
- $\omega \geq \frac{\alpha^2}{2}$, $c = \pm\alpha\sqrt{2\omega - \alpha^2}$.

Idea of proof

1. For all $\{g_0, g_1\}$ whose asymptotics is associated with $\{\alpha, \omega, c\}$, where $c = c_1 + \mathrm{i}c_2$, the t-equation of the Lax pair for the NLS (at $x = 0$) has a solution $\Phi(t, k)$, $k \in \Sigma$, s.t.

   $\Phi(t, k) = \Psi(t, k)(1 + o(1))$ as $t \to +\infty$, where

$$\Psi(t, k) = \mathrm{e}^{\mathrm{i}\omega t\sigma_3} E(k) \mathrm{e}^{-\mathrm{i}\Omega(k)t\sigma_3}, \quad \Gamma = \{k : \operatorname{Im}\Omega(k) = 0\},$$

$$\Omega^2(k) = k^4 + 4\omega k^2 - 4\alpha c_2 k + (\alpha^2 - \omega)^2 + c_1^2 + c_2^2.$$

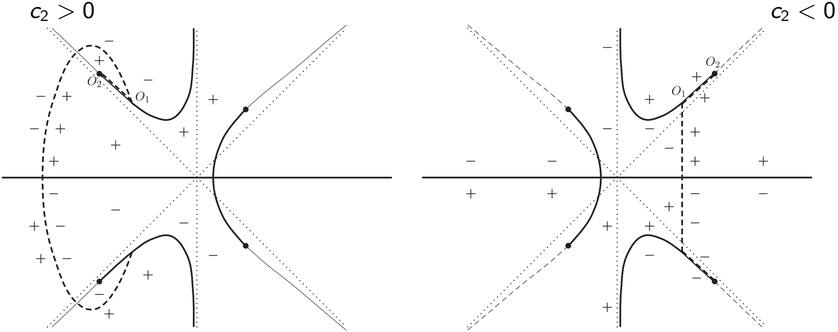
2. $\Sigma = \Gamma \cup \{\text{branch cuts}\}$ is the contour for the RH problem for the inverse spectral mapping $\{A(k), B(k)\} \to \{g_0, g_1\}$.

3. Compatibility of $\{q_0, g_0, g_1\}$ in spectral terms: global relation

$$A(k)b(k) - a(k)B(k) = 0, \quad k \in D = \{k : \operatorname{Im} k \geq 0, \operatorname{Im}\Omega(k) \geq 0\}.$$

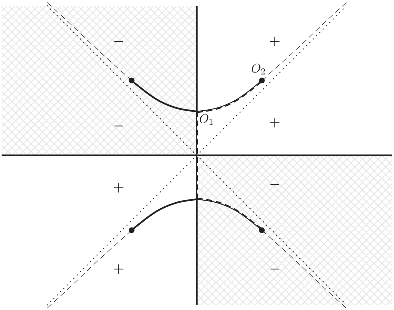
4. Existence of a (finite) arc of $\Sigma_0 = \Gamma \cap \{\text{branch cuts}\}$ in D contradicts the global relation (particularly, the continuity of $b(k)$ and $a(k)$ across the arc).

Non-admissible spectral curves: $\omega > 0$, I

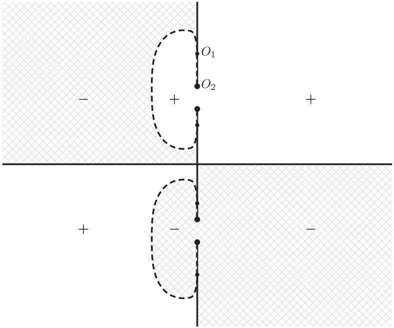


Non-admissible spectral curves: $\omega > 0$, II

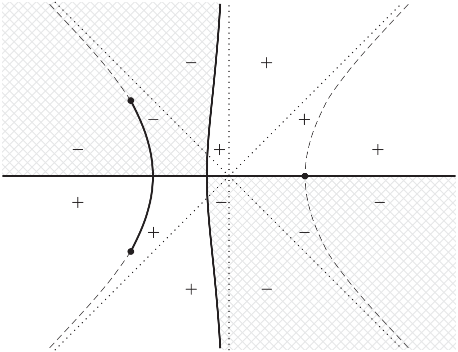
$c_2 = 0,\ 0 < \omega < \frac{\alpha^2}{2}$



$c_2 = 0,\ c_1^2 < \alpha^2(2\omega - \alpha^2)$



Admissible spectral curves: $\omega < 0$



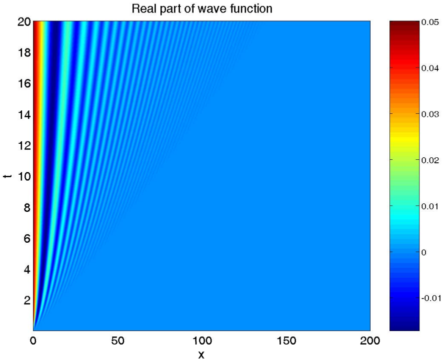
Range $\omega < 0$, $c_2 > 0$: the only admissible case is when the finite arc of $\{\mathrm{Im}\,\Omega(k) = 0\}$ lying on the right branch of the curve $\{\mathrm{Im}\,\Omega^2(k) = 0\}$ degenerates to a point on $\mathbb{R}$, i.e., when $\Omega^2(k)$ has a double, positive zero. In terms of $\{\alpha, \omega, c\}$, this corresponds to:

$$c_1 = 0, \; c_2 = \alpha\sqrt{2(\alpha^2 - \omega)}.$$

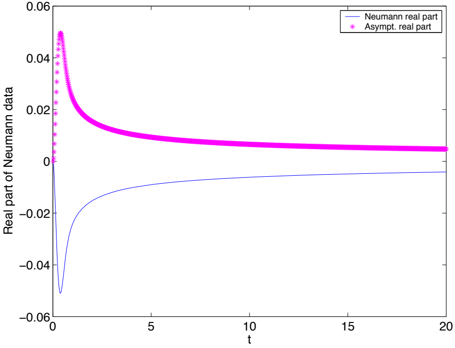
Numerics for $-3\alpha^2 < \omega < \alpha^2/2$, II

$\alpha = 0.05, \quad \omega = 0$

$q_0(x) \equiv 0, \quad g_0(t) = \alpha + \mathrm{O}(\mathrm{e}^{-10t^2})$



Real part of $q(x,t)$

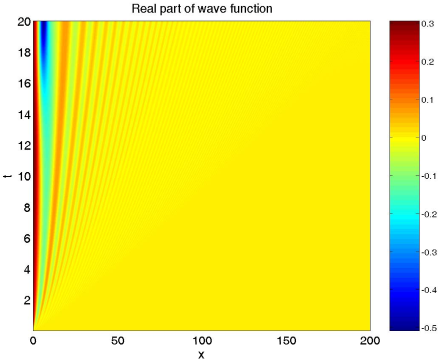


Neumann data

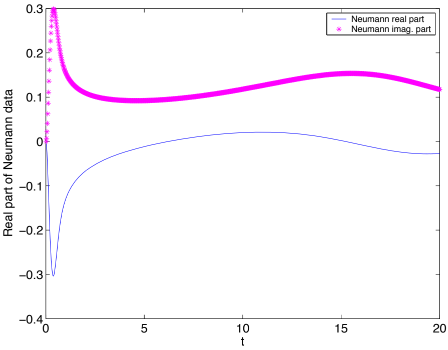
Numerics for $-3\alpha^2 < \omega < \alpha^2/2$, III

$\alpha = 0.3, \quad \omega = 0$

$q_0(x) \equiv 0, \;\; g_0(t) = \alpha + \mathrm{O}(\mathrm{e}^{-10t^2})$



Real part of $q(x,t)$

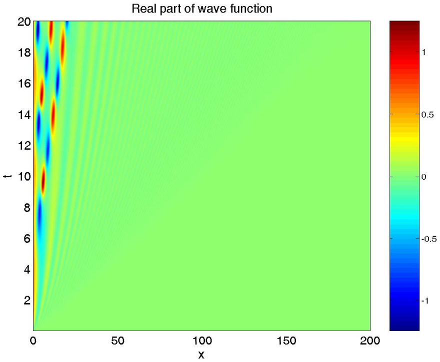


Neumann data

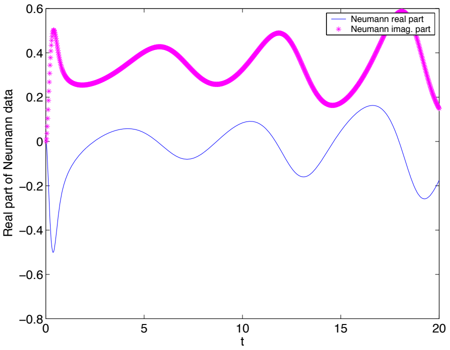
Numerics for $-3\alpha^2 < \omega < \alpha^2/2$, IV

$\alpha = 0.5, \quad \omega = 0$

$q_0(x) \equiv 0, \;\; g_0(t) = \alpha + \mathrm{O}(\mathrm{e}^{-10t^2})$



Real part of $q(x,t)$

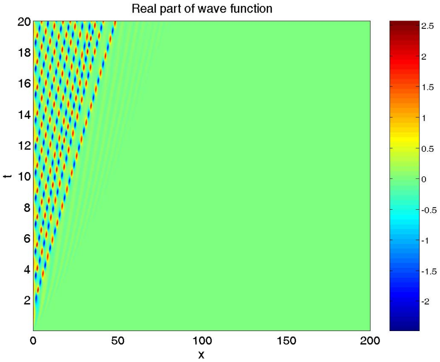


Neumann data

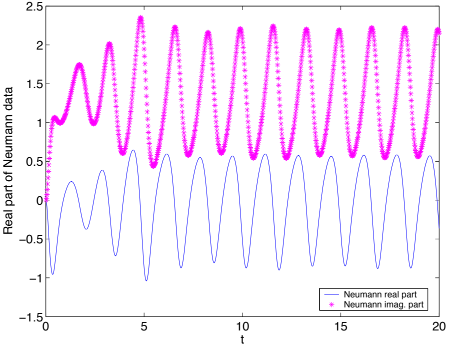
Numerics for $-3\alpha^2 < \omega < \alpha^2/2$, V

$\boxed{\alpha = 1, \quad \omega = 0}$

$q_0(x) \equiv 0, \quad g_0(t) = \alpha + \mathrm{O}(\mathrm{e}^{-10t^2})$



Real part of $q(x,t)$



Neumann data

References I

A.S.Fokas.
Integrable nonlinear evolution equations on the half-line,
Comm. Math. Phys. **230**, (2002) 1–39.

A. Boutet de Monvel, V. Kotlyarov.
Generation of asymptotic solitons of the nonlinear Schrödinger equation by boundary data,
J. Math. Phys. 44, 8 (2003) 3185–3215.

A. Boutet de Monvel, A.S.Fokas, D. Shepelsky.
Analysis of the global relation for the nonlinear Schrödinger equation on the half-line,
Lett. Math. Phys. **65** (2003), 199–212.

A. Boutet de Monvel, A.S.Fokas, D. Shepelsky.
The modified KdV euation on the half-line,
J. Inst. Math. Jussieu, **3**, (2004) 139–164.

A.S. Fokas, A. Its.
The nonlinear Schrödinger equation on the interval,
J.Phys.A: Math. Gen. **37**, (2004) 6091–6114.

References II

A. Boutet de Monvel, V. Kotlyarov.
Focusing nonlinear Schrödinger equation on the quarter plane with time-periodic boundary condition: a Riemann-Hilbert approach,
J. Inst. Math. Jussieu, **6**, 4 (2007) 579–611.

A. Boutet de Monvel, A.S.Fokas, D. Shepelsky.
Integrable nonlinear evolution equations on a finite interval,
Commun. Math. Phys. **263** (2006), 133–172.

A.S.Fokas.
A unified approach to boundary value problems.
CBMS-NSF Regional Conference Series in Applied Mathematics, 78. SIAM, Philadelphia, PA, 2008

A. Boutet de Monvel, A. Its, V. Kotlyarov.
Long-time asymptotics for the focusing NLS equation with time-periodic boundary condition on the half-line,
Comm. Math. Phys. **290** (2009), 479–522.

References III

A. Boutet de Monvel, V. Kotlyarov, D. Shepelsky
Decaying long-time asymptotics for the focusing NLS equation with periodic boundary condition,
International Mathematics Research Notices, No. 3 (2009), 547–577.

A. Boutet de Monvel, D. Shepelsky.
Long time asymptotics of the Camassa–Holm equation on the half-line,
Ann. Inst. Fourier **59**, (2009), 3015–3056.

J. Lenells, A. S. Fokas.
The unified method: II. NLS on the half-line with t-periodic boundary conditions.
J. Phys. A **45** (2012), 195202.

A. Its, D. Shepelsky.
Initial boundary value problem for the focusing nonlinear Schrödinger equation with Robin boundary condition: half-line approach,
Proc. R. Soc. A **469** (2013), 20120199 (15pp).