Quantitative unique continuation principles and application to control theory and random Schrödinger operators

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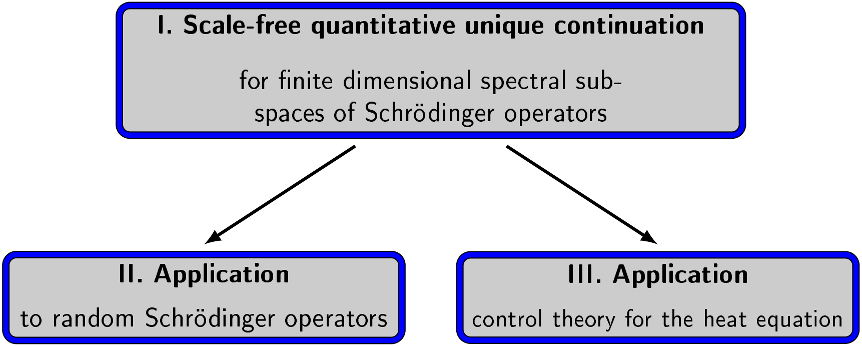
(joint work with I. Nakić, M. Täufer and I. Veselić)

Technische Universität Chemnitz

September 13, 2016

German-Russian-Ukrainian summer school on Spectral Theory, Differential Equations and Probability

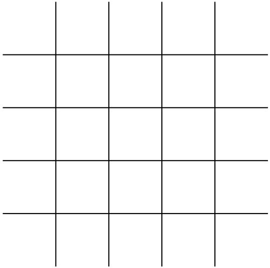
Contents

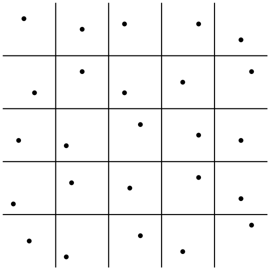


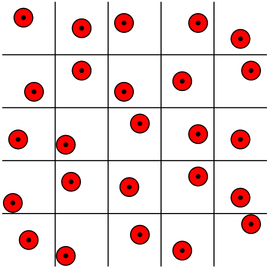
I. Nakić, M. Täufer, M.T., I. Veselić. *Scale-free unique continuation principle, eigenvalue lifting and Wegner estimates for random Schrödinger operators*, arXiv:1609.01953 [math.AP], 2016.

I. Nakić, M. Täufer, M.T., I. Veselić. *Scale-free uncertainty principles and Wegner estimates for random breather potentials*, C. R. Math. 353(10):919-923, 2015.

I. Scale-free quantitative unique continuation







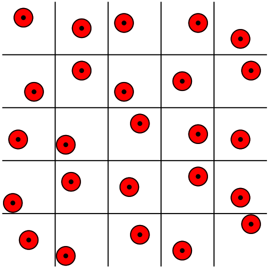
Definition

Let $\delta \in (0, 1/2)$. A sequence $(z_j)_{k \in \mathbb{Z}^d}$ is called δ-equidistributed if

$$\forall j \in \mathbb{Z}^d : \quad B(\delta, z_j) \subset \Lambda_1 + j,$$

where

$$\Lambda_L = (-L/2, L/2)^d.$$



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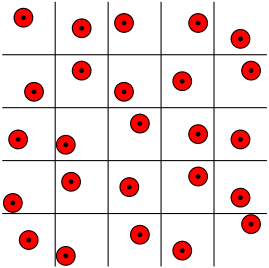

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For $L \in \mathbb{N}$ we define

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Note that $W_\delta(L)$ depends on $(z_j)_j$!

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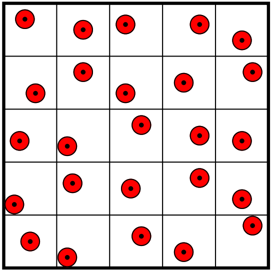

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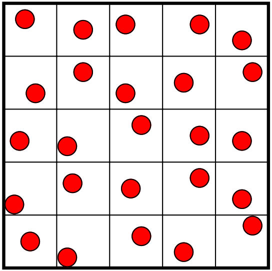
Let $V \in L^\infty(\mathbb{R}^d)$ and consider

$$H_L = -\Delta + V \quad \text{in } L^2(\Lambda_L)$$

with Dirichlet, Neumann or periodic b.c.



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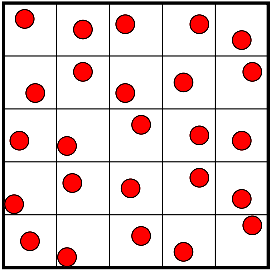

- H_L lower bounded, purely discrete spectrum
- $\phi \in \mathrm{Ran}(\chi_{(-\infty,b]}(H_L)) \quad \Leftrightarrow \quad \phi = \sum_{k \in \mathbb{N}, E_k \leq b} \alpha_k \phi_k$

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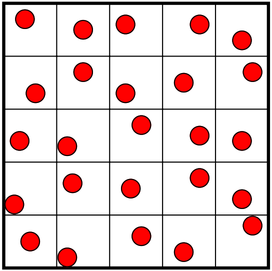

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Theorem (Nakić, Täufer, T., Veselić)

$$\|\phi\|^2_{L^2(\Lambda_L)} \leq C_{\mathrm{sfuc}} \|\phi\|^2_{L^2(W_\delta(L))}, \quad \text{where} \quad C_{\mathrm{sfuc}} = \delta^{-N(1+\|V\|_\infty^{2/3}+|b|^{1/2})}.$$

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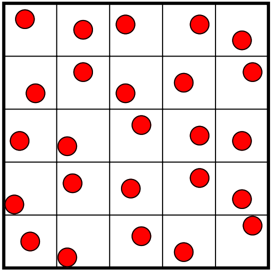

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Theorem (Nakić, Täufer, T., Veselić) There exists $N = N(d) > 0$ s.t.

- for all $\delta \in (0, 1/2)$ and $b > 0$
- for all δ-equidistributed sequences $(z_j)_j$,
- all $L \in \mathbb{N}$

and all $\phi \in \mathrm{Ran}(\chi_{(-\infty,b]}(H_L))$ we have

$$\|\phi\|^2_{L^2(\Lambda_L)} \leq C_{\mathrm{sfuc}} \|\phi\|^2_{L^2(W_\delta(L))}, \quad \text{where} \quad C_{\mathrm{sfuc}} = \delta^{-N(1+\|V\|_\infty^{2/3}+|b|^{1/2})}.$$

II. Application to random Schrödinger operators

Consider family of Schrödinger operators

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$$\boxed{\begin{matrix} u = \chi_{B_1} \\ \omega_k \sim \mathcal{U}[0, \omega_+],\ \omega_+ < 1/2 \end{matrix}} \quad \Rightarrow \quad \boxed{V_\omega^{\mathrm{B}}(x) = \sum_{k \in \mathbb{Z}^d} \chi_{B_{\omega_j}}(x-k)}$$

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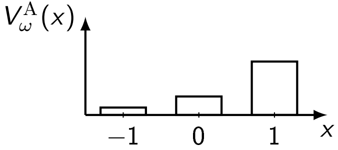

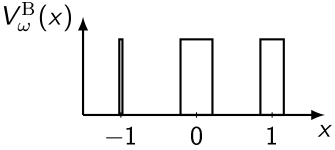

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$\exists \Sigma \subset \mathbb{R}$ such that $\sigma(H_\omega) = \Sigma$ for almost all $\omega \in \Omega$.

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Phenomenon of localization

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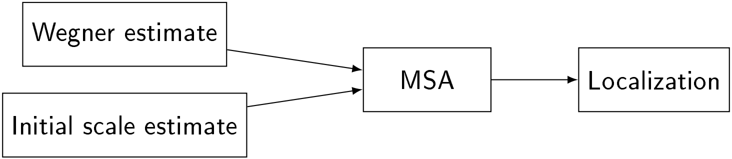
Dynamical localization

There are intervals $I \subset \Sigma$, such that for every $\psi_0 \in \mathcal{H}$, all $p \geq 0$ and almost all $\omega \in \Omega$

$$\sup_{t \in \mathbb{R}} \left\| |x|^p \mathrm{e}^{-\mathrm{i} H_\omega t} \chi_I(H_\omega) \psi_0 \right\| < \infty.$$

Multiscale analysis [FS83], [FMSS85]



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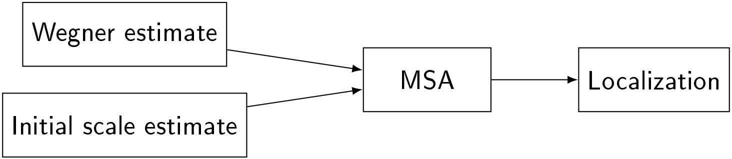

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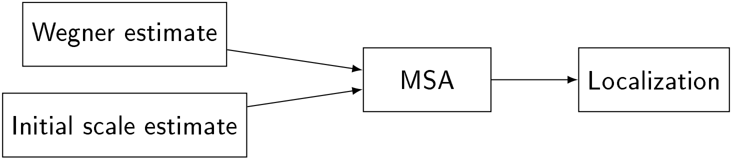

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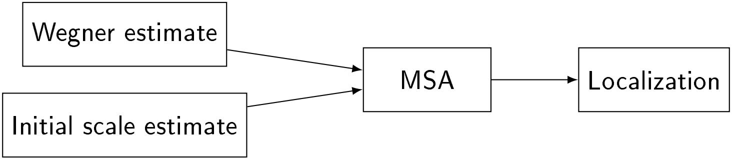

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$$V_\omega^{\mathrm{A}}(x) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x-k)$$
- non-linear (and monotone) in the breather model
$$V_\omega^{\mathrm{B}}(x) = \sum_{k \in \mathbb{Z}^d} u\left(\frac{x-k}{\omega_k}\right).$$

What is a Wegner estimate?

- $\Lambda_L = [-L/2, L/2]^d$ and $H_{\omega,L}$ on $L^2(\Lambda_L)$ with Dirichlet b.c.

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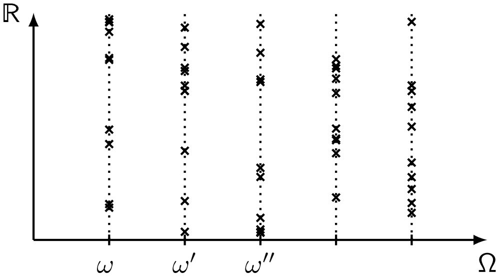

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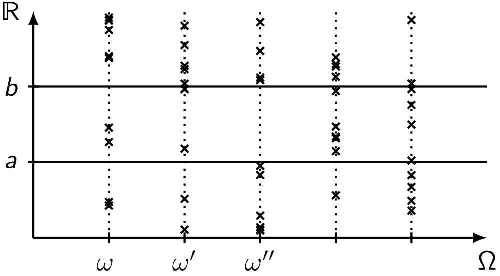

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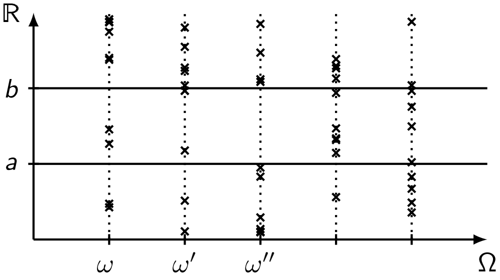

- Wegner estimate [Weg81] is upper bound on expected number of eigenvalues of $H_{\omega,L}$ in intervall $[a, b]$:

$$\forall L \in \mathbb{N} \text{ and } [a,b] \subset \mathbb{R} : \quad \mathbb{E}\big(\operatorname{Tr} \chi_{[a,b]}(H_{\omega,L})\big) \leq C_{\mathrm{W}}(b-a)^s |\Lambda_L|^m$$

with $C_{\mathrm{W}} > 0$ und $s \in (0,1]$ and $m \in [1, \infty)$.

Theorem [Nakić, Täufer, T., Veselić 15]

Wegner estimate for random breather model with $u = \chi_B$.
(generalizes to a large class of single-site potentials)

Earlier results for breather model

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- [Combes Hislop Mourre 96] prove a Wegner estimate assuming $u \in C^2$, $u \geq 0$, compactly supported,

$$-\langle x, \nabla u \rangle \geq 0 \quad \text{and} \quad \exists c_0 > 0 : \left| \frac{\langle x, \mathrm{Hess}[u] x \rangle}{\langle x, \nabla u \rangle} \right| \leq c_0 < \infty.$$

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- first condition is decay assumpiton
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- this condition implies that u has a singularity at origin
- If we take for example $u(x) = |x|^{-\alpha}$, then

$$u\left(\frac{x}{\omega_j}\right) = \omega_j^{\alpha} |x|^{-\alpha} = \omega_j^{\alpha} u(x)$$

random breather model $\backsimeq$ alloy-type model

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Idea of proof

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Idea of proof

An important step to prove a Wegner estimate

$$\forall L \in \mathbb{N} \text{ and } [a,b] \subset \mathbb{R} : \quad \mathbb{E}\big(\operatorname{Tr} \chi_{[a,b]}(H_{\omega,L})\big) \leq C_{\mathrm{W}} (b-a)^s |\Lambda_L|^m$$

is to show that the eigenvalues move, i.e.

$$\lambda_i\big(H_{\omega+\delta,L}\big) \geq \lambda_i\big(H_{\omega,L}\big) + f(\delta)!$$

Eigenvalue lifting

$$\lambda_i\big(H_{\omega+\delta,L}\big) \geq \lambda_i\big(H_{\omega,L}\big) + f(\delta), \quad W := V_{\omega+\delta} - V_\omega$$

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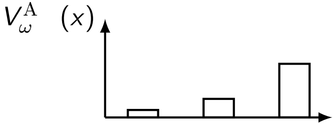

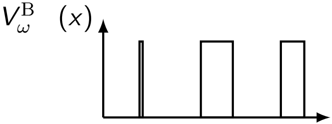

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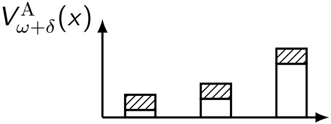

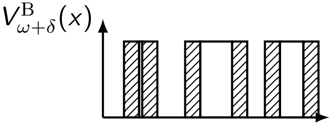

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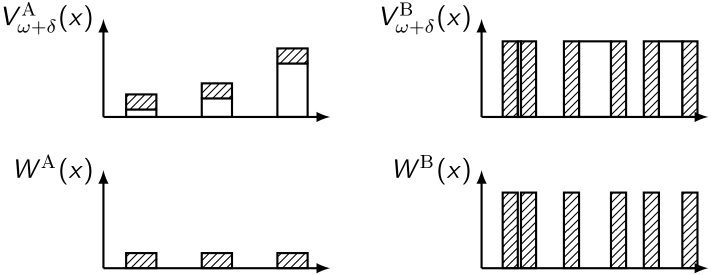

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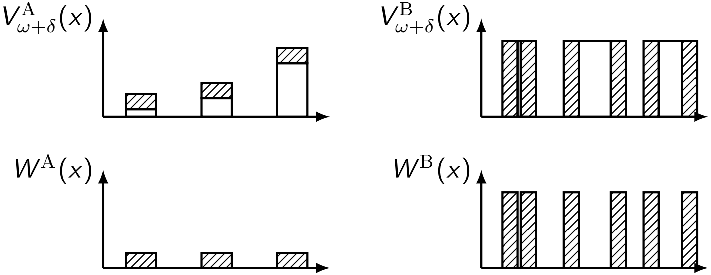

periodic
ω-independent

non-periodic
ω-dependent

Eigenvalue liftung

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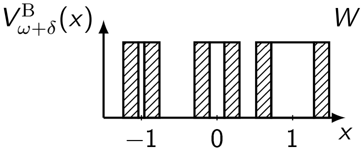

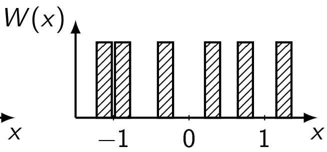

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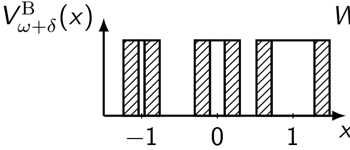

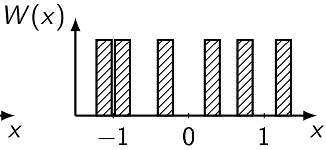

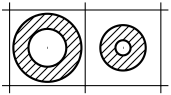

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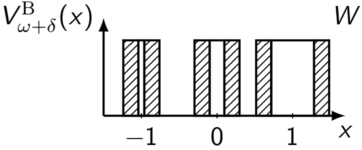

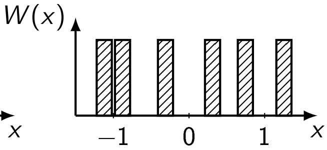

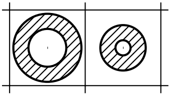

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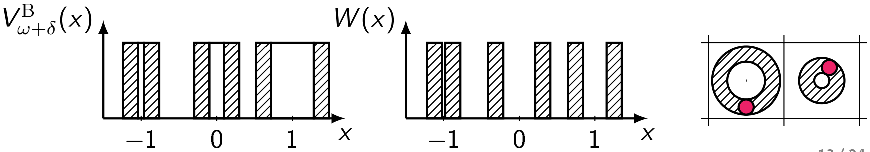

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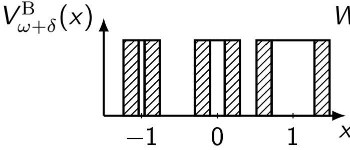

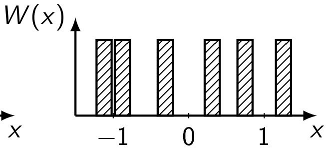

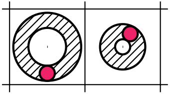

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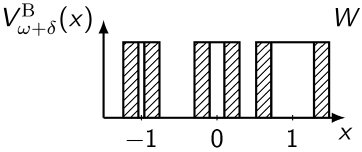

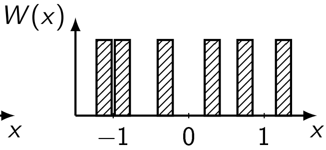

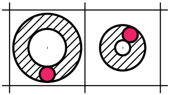

Theorem [Nakić, Täufer, T., Veselić 15]

Let

- $b, \alpha > 0$, $\delta \in (0, 1/2)$, $L \in \mathbb{N}$
- $A, B : \Lambda_L \to \mathbb{R}$ be measurable and bounded, and

$$B \geq \alpha \chi_{W_\delta(L)}$$

for some δ-equidistributed sequence.

Then for all $i \in \mathbb{N}$ with $\lambda_i(-\Delta + A + B) \leq b$, we have

$$\lambda_i(-\Delta_L + A + B) \geq \lambda_i(-\Delta_L + A) + \alpha \delta^{N_d(1 - \|A+B\|_\infty^{2/3} + \sqrt{b})}.$$

III. Control theory for the heat equation

Heat equation in $\Omega \subset \mathbb{R}^d$ with control $f \in L^2(\Omega \times [0, T])$ on $S \subset \Omega$:

$$\begin{cases} \partial_t y - \Delta y + Vy = \mathbf{1}_S f & \text{in } \Omega \times [0, T] \\ y = 0 & \text{on } \partial\Omega \times [0, T] \\ y(0) = y_0 & y_0 \in L^2(\Omega) \end{cases} \tag{1}$$

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Example [Fernandez-Cara & Münch 14]: $\Omega = (0, 1)$, $S = (0.3, 0.6)$, $T = 0.5$

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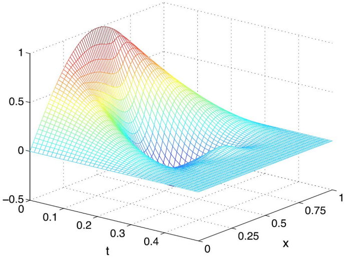

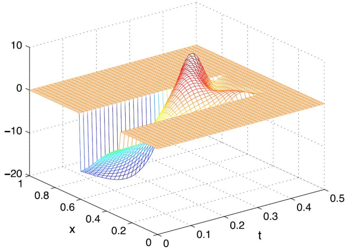

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Our contribution:

- explicit dependence of $\mathcal{C}$ on the geometry

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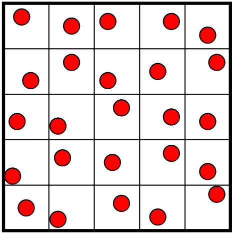

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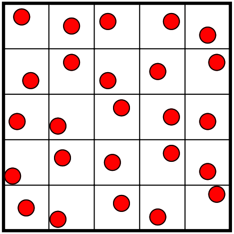

Theorem (Nakić, Täufer, T., Veselić 15)

$$\mathcal{C} \leq C_1 \delta^{-C_2(1+\|V\|_\infty^{2/3})} \cdot \mathrm{e}^{2\|V\|_\infty + C_3/T} \|y_0\|_{L^2(\Lambda_L)}.$$

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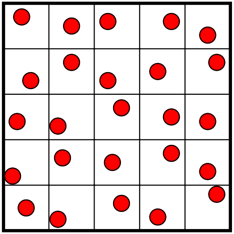

Theorem (Nakić, Täufer, T., Veselić 15) For any $\delta \in (0, 1/2)$, any $V \in L^\infty(\mathbb{R}^d)$, any $L \in \mathbb{N}$, any δ-equidistributed sequence, and any $y_0 \in L^2(\Lambda_L)$ the system is Null-controllable with cost

$$\mathcal{C} \leq C_1 \delta^{-C_2(1+\|V\|_\infty^{2/3})} \cdot \mathrm{e}^{2\|V\|_\infty + C_3/T} \|y_0\|_{L^2(\Lambda_L)}.$$

Outline of the proof:

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- Original system

$$\begin{cases} \partial_t y - \Delta y + Vy = \mathbf{1}_{W_\delta(L)} f & \text{in } \Lambda_L \times [0, T] \\ y = 0 & \text{on } \partial\Lambda_L \times [0, T] \\ y(0) = y_0 & y_0 \in L^2(\Lambda_L) \end{cases} \quad (1)$$

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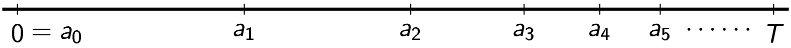

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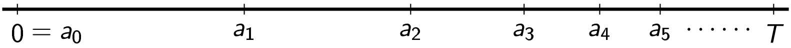

- On each interval construct control function satisfying

$$\|y(\cdot, a_{j+1})\|_{L^2(\Lambda_L)} \leq C_j \|y(\cdot, a_j)\|_{L^2(\Lambda_L)}, \quad \prod_{j=1}^{n} C_j \to 0, \quad n \to \infty$$

Denote $P_E = \chi_{(-\infty,E]}(-\Delta)$ and consider the problem

$$\begin{cases} \partial_t y - \Delta y = P_E \mathbf{1}_{W_\delta(L)} g & \text{in } \Lambda_L \times [0,t] \\ y = 0 & \text{in } \partial\Lambda_L \times [0,t] \\ y(\cdot,0) = y_0 & y_0 \in \operatorname{Ran} P_E \end{cases} \tag{2}$$

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$$\|z(t,\cdot)\|^2_{L^2(\Lambda_L)} \le \kappa_t \|z\|^2_{L^2(W_\delta(L)\times[0,t])}.$$

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- Moreover $\mathcal{C} = \mathcal{C}(t, y_0) \le \sqrt{\kappa_t}\|y_0\|_{L^2(\Lambda_L)}$.

Final state observability of (3)

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\begin{aligned}
t\|z(\cdot,t)\|^2_{L^2(\Lambda_L)} &= \int_0^t \|z(\cdot,t)\|^2_{L^2(\Lambda_L)}\mathrm{d}s \\
&= \int_0^t \|\mathrm{e}^{(\Delta)(t-s)} z(\cdot,s)\|^2_{L^2(\Lambda_L)}\mathrm{d}s \\
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Since $z_0 \in \operatorname{Ran} P_E$ we have $z(\cdot,s) \in \operatorname{Ran} P_E$. Unique continuation implies

$$
\begin{aligned}
t\|z(\cdot,t)\|^2_{L^2(\Lambda_L)} &\leq C_{\mathrm{sfuc}}(E) \int_0^t \|z(\cdot,s)\|^2_{L^2(W_\delta(L))} \mathrm{d}s \\
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We have shown Final state observability with

$$
\kappa_t = \delta^{-N(1+\|V\|_\infty^{2/3}+|E|^{1/2})} t^{-1}.
$$

Using the equivalence between controllability and observability we obtain that the system

$$\begin{cases} \partial_t y - \Delta y = P_E \mathbf{1}_S g & \text{in } \Lambda_L \times [0,t] \\ y = 0 & \text{in } \partial\Lambda_L \times [0,t] \\ y(\cdot,0) = y_0 & y_0 \in \operatorname{Ran} P_E \end{cases} \tag{2}$$

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In particular, for all $E \in \mathbb{R}$, $t > 0$ and $y_0 \in \operatorname{Ran} P_E$ there is

$$g = g(E, t, y_0)$$

such that

$$y(\cdot, t) \equiv 0$$

and

$$\|g\|_{L^2(W_\delta(L)\times(0,t))} \leq \sqrt{C_{\mathrm{sfuc}}(E)}\, t^{-1/2} \|y_0\|_{L^2(\Lambda_L)}$$

where

$$C_{\mathrm{sfuc}} = \delta^{-N(1+\|V\|_\infty^{2/3}+|E|^{1/2})}$$

Construction of control function f for the system

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\begin{cases}
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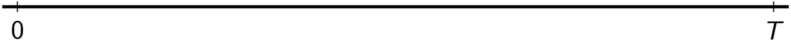

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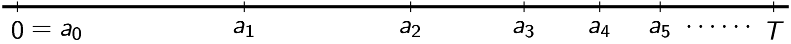

Decomposition:

$$[0, T] = \bigcup_{j=0}^{\infty} [a_j, a_{j+1}], \quad a_0 = 0, \quad a_{j+1} = a_j + 2T_j, \quad \sum_{j=0}^{\infty} 2T_j = T$$

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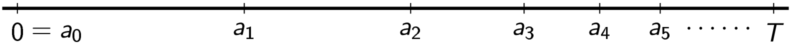

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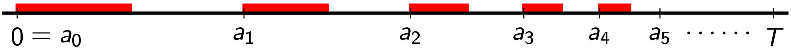

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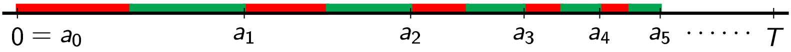

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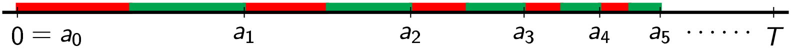

Active phase: We choose $E_j = 2^{2j}$ and $f = g(E_j, T_j, P_{E_j} y(\cdot, a_j))$.

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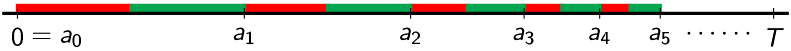

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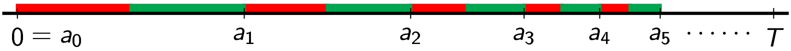

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$$\|y(a_{j+1})\|_{L^2(\Lambda_L)} \le \mathrm{e}^{-T_j E_j} \|y(a_j + T_j)\|_{L^2(\Lambda_L)}.$$

Putting together the acive and passive phases we obtain

$$\begin{aligned}\|y(\cdot, a_{j+1})\|_{L^2(\Lambda_L)} &\leq \prod_{k=0}^{j} \mathrm{e}^{-T_k E_k} \left(1 + \sqrt{C_{\mathrm{sfuc}}(E_k)}\right) \|y_0\|^2_{L^2(\Lambda_L)} \\ &\to 0 \quad \text{if } j \to \infty.\end{aligned}$$

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$\|f\|_{L^2(\Lambda_L \times [0,T])}$

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Thank you for your attention!