Distribution of eigenvalues of sample covariance matrices with tensor product samples

D. Tieplova

V. N. Karazin Kharkiv National University, Kharkiv, Ukraine

12. 09. 2016

Outline

- Classical hermitian and real symmetric models.
- Global regime: NCM and CLT for LES.
- Main tools and approaches.
- Previous results.
- The problem and main result.
- The idea of proof.

Introduction

Random matrices was initiated in the 1920s– 1930s by statisticians and introduced in physics in the 1950s– 1960s by Wigner and Dyson. They widely used in fields such:

- quantum field theory, quantum mechanics (quantum chaos)
- probability theory, statistics, statistical mechanics
- telecommunication theory
- combinatorics, operator theory, number theory
- etc...

Examples

Wigner ensemble [W:57]

M - hermitian or real symmetric $n \times n$ matrices

$$M_{ij} = n^{-1/2} W_{ij},$$

$\{W_{ij}\}_{1 \le i \le j \le n}$-i.i.d.random variables

$$E\{W_{ij}\} = 0,\ E\{|W_{ij}|^2\} = 1,\ E\{|W_{ii}|^2\} = 2,$$

Deformed Wigner ensemble [P:72]

$$M_{ij} = H^{(0)} + n^{-1/2} W_{ij},$$

where $H^{(0)}$ some random or non random matrix independent on W.

Sample covariance ensemble [MP:67]:

$$M = n^{-1}B(YY^*)B,$$

Y – real or complex $n \times m$ matrix with i.i.d. entries Y_{ij}

$$E\{Y_{ij}\} = 0, \quad E\{|Y_{ij}|^2\} = 1$$

B–$n \times n$ positive matrix bounded uniformly in n and independent of Y

Deformed sample covariance ensembles

$$M = H^{(0)} + n^{-1}B(Y + A)(Y + A)^*B,$$

where $H^{(0)}$, A and $B > 0$ do not depend on Y.

Global regime: main objects and problems

Let $\{\lambda_i\}_{i=1}^n$ be random eigenvalues of our M.

Normalized Eigenvalue Counting Measure

$$N_n[\Delta] = \sharp\{\lambda_i \in \Delta\}/n$$

Linear Eigenvalue Statistics corresponding to the test function h

$$\mathcal{N}_n[h] = \sum_{i=1}^{n} h(\lambda_i) = \int h(\lambda) N_n(d\lambda)$$

Problems:

- Find $\lim_{n\to\infty} E\{N_n[\Delta]\}$ or $\lim_{n\to\infty} E\{n^{-1}\mathcal{N}_n[h]\}$;
- Prove that $Var\{n^{-1}\mathcal{N}_n[h]\} \to 0$ as $n \to \infty$ and find the rate of convergence;
- Find $Cov\{\mathcal{N}_n[h_1], \mathcal{N}_n[h_2]\}$;
- Prove CLT for the fluctuation of $\mathcal{N}_n[h]$ for smooth test functions;
- Prove CLT for the fluctuation of $\mathcal{N}_n[h]$ for indicators;

Main approaches to the global regime: the moment and the resolvent one

Both approaches are based on the simple formula:

$$\mathcal{N}_n[h] = \mathrm{Tr}\, h(M)$$

Moment approach

Set $h_k(\lambda) = \lambda^k$, then

$$\mathcal{N}_n[h_k] = \mathrm{Tr}\, M^k, \quad E\{\mathcal{N}_n[h_k]\} = E\{\mathrm{Tr}\, M^k\}$$

$$Cov\{\mathcal{N}_n[h_k], \mathcal{N}_n[h_m]\} = Cov\{\mathrm{Tr}\, M^k, \mathrm{Tr}\, M^m\}$$

$$E\{\mathrm{Tr}\, M^k\} = \int \lambda^k N_n(d\lambda)$$

So if we want to find the limit of $E\{\mathcal{N}_n[h]\}$ it is sufficient to find limits of $E\{\mathrm{Tr}\, M^k\}$.

Historically the moment approach was the first one. But the resolvent approach provides more simple proofs.

Resolvent approach

Consider any $z : \Im z \neq 0$ and set $h_z(\lambda) = (\lambda - z)^{-1}$, then

$$\mathcal{N}_n[h_z] = \mathrm{Tr}(M - z)^{-1}, \quad E\{\mathcal{N}_n[h_z]\} = E\{\mathrm{Tr}(M - z)^{-1}\}$$

$$Cov\{\mathcal{N}_n[h_{z_1}], \mathcal{N}_n[h_{z_2}]\} = Cov\{\mathrm{Tr}\,(M - z_1)^{-1}, \mathrm{Tr}\,(M - z_2)^{-1}\}$$

The Stieltjes transform f_n of N_n,

$$f_n(z) = \int \frac{N_n(d\lambda)}{\lambda - z}, \quad \Im z \neq 0.$$

$$f_n(z) = n^{-1} E\{\mathrm{Tr} G_M(z)\}, \quad G_M(z) = (M - z)^{-1}$$

There is the one-to-one correspondence between finite nonnegative measures and their Stieltjes transforms, so if we can find the limit of Stieltjes transforms f_n then we can find the limiting measure N.

Zero level results: law of large numbers for LES

The sample covariance matrices can be written as follows

$$M = \sum_{\mu=1}^{m} B(Y^\mu \otimes Y^\mu)B.$$

In this case we can use the well-know formula

$$G_{M+Y\otimes\bar{Y}} = G_M - \frac{G_M(Y\otimes\bar{Y})G_M}{1+(G_M Y, Y)}.$$

It is naturally to expect that $G_{M+Y\otimes\bar{Y}}$ and G_M are not very different. So with this formula we can find the equation on $E\{\mathrm{Tr}G\}$.
Commonly it convenient to use the similar identity

$$G_{pp} = -\frac{1}{z + M_{pp} + (G^{(p)}m^{(p)}, m^{(p)})},$$

where $G^{(p)} = (M^{(p)} - z)^{-1}, \quad m^{(p)} = (M_{p,1}, \ldots, M_{p,p-1}, M_{p,p+1}, \ldots, M_{p,n})$, and $M^{(p)}$ denotes M without the p-th line and the p-th column.

Wigner ensembles theorems [W:58]

$$M_n = n^{-1/2} W_n$$

Denote by N_n the NCM of eigenvalues of W_n. Then $N_n \xrightarrow{w} N$, where N is non-random probability measure, and for the Stieltjes transform f of N we have:

$$f(z) = \frac{-z + \sqrt{z^2 - 4}}{2}.$$

Deformed semicircle law [P:72]

$$M_n = H_n^{(0)} + n^{-1/2} W_n$$

Assume that the NCM $N_n^{(0)}$ of $H^{(0)}$ converges weakly to a nonnegative probability measure $N^{(0)}$(with the Stieltjes transform $f^{(0)}$). Then

$$f(z) = f^{(0)}(z + f(z)).$$

Marchenko-Pastur theorem [MP:67]

$$M_n = n^{-1}B(YY^*)B.$$

Denote by N_n and σ_n the Normalized Counting Measure of eigenvalues of M_n and B^2 respectively. Assume that

$$\lim_{n\to\infty} \sigma_n = \sigma,$$
$$m_n \to +\infty,\ n \to +\infty,\ c_n = m_n/n \to c \in [0, +\infty).$$

Then N_n converge weakly in probability to a non-random probability measure N. And the Stieltjes transform f of N is uniquely determined by the equation

$$f(z) = \left(\int \frac{\tau\sigma(d\tau)}{1+\tau f(z)} - z \right)^{-1}$$

in the class of Stieltjes transforms of probability measures.

Previous results

Wigner ensemble

- L.Pastur(1972):
  The convergence of the Normalized Counting Measures of W under the minimal conditions on the distribution of w_{ij} (the Lindeberg type conditions);
- A.Khorunzhy, B.Khoruzhenko, L.Pastur (1996):
  Covariance of traces of resolvents for Wigner matrices;
- Z.Bai, J.W.Silverstein (2004):
  CLT for polynomial test functions for some generalizations of the Wigner and sample covariance matrices with $B \neq I$ with $\kappa_4 = 0$;
- A.Lytova, L. Pastur (2009):
  CLT for Wigner and sample covariance matrix with $B = I$, $\kappa_4 \neq 0$ and test functions possessing 5 derivatives;
- M.Shcherbina (2011):
  CLT for Wigner and sample covariance matrix with $B = I$, $\kappa_4 \neq 0$ and test functions possessing 2 derivatives.

Sample covarience matrices

- J.W.Silverstein, R.B.Dozier (2004):
  The convergence of the Normalized Counting Measures of
  $H = \frac{1}{N}(R_n + \sigma X_n)(R_n + \sigma X_n)^*$;
- Z.Bai, J.W.Silverstein (2004):
  CLT for sample covariance matrix with $B \neq I$ with $\kappa_4 = 0$ and analytic test functions;
- A.Lytova, L. Pastur (2009):
  CLT for Wigner and sample covariance matrix with $B = I$, $\kappa_4 \neq 0$ and test functions possessing 5 derivatives;
- M.Shcherbina (2011):
  CLT for Wigner and sample covariance matrix with $B = I$, $\kappa_4 \neq 0$ and test functions possessing 2 derivatives;
- J.Najim, J.Yao (2014):
  CLT for sample covariance matrix with $B \neq I$ and test functions possessing 5 derivatives.

Our model

Consider hermitian random matrices:

$$M_n = \frac{1}{n^2}\sum_{\mu=1}^{m} X^\mu \otimes \bar{X}^\mu,$$

where $X^\mu = B(Y^\mu \otimes Y^\mu)$(in standard model $X^\mu = BY^\mu$) and $\{Y^\mu\}_{\mu=1}^m$ are i.i.d. random vectors from $\mathbb{C}^n$, such that:

$$E\{Y_i^\mu\} = E\{Y_i^\mu Y_k^\nu\} = 0,\ E\{Y_i^\mu \bar{Y}_k^\mu\} = \delta_{ik}.$$

$B = \{B_{\bar{i},\bar{j}}\}$ is an $n^2 \times n^2$ hermitian matrix, where $\bar{i} = (i_1, i_2)$ $\bar{j} = (j_1, j_2)$ are multi-indexe.
Introduce the $n^2 \times n^2$ matrix

$$J_{\bar{p},\bar{q}} = \delta_{\bar{p}\,\bar{q}} + \delta_{\bar{p}'\,\bar{q}},$$

where $\bar{p}' = (p_2, p_1)$.

Theorem

Denote by N_n and σ_n the Normalized Counting Measure of eigenvalues of M_n and BJB respectively. Assume that

$$\lim_{n\to\infty} \sigma_n = \sigma,$$
$$m_n \to +\infty,\ n \to +\infty,\ c_n = m_n/n^2 \to c \in [0, +\infty).$$

Then N_n converge weakly in probability to a non-random probability measure N. And if $f^{(0)}$ is the Stieltjes transform of σ, then the Stieltjes transform f of N is uniquely determined by the equation

$$f(z) = f^{(0)}\left(\frac{z}{c - zf(z) - 1}\right)(c - zf(z) - 1)^{-1}$$

in the class of Stieltjes transforms of probability measures.

The idea of proof (1)

We use the resolvent method. So denote

$$G = (M_n - z)^{-1},\ G^\mu = G\,|_{X^\mu = 0}\ .$$

In our case:

$$G = G^\mu - n^{-2}\frac{G^\mu (X^\mu \otimes \bar{X}^\mu) G^\mu}{1 + n^{-2}(G^\mu X^\mu, X^\mu)}.$$

This imply that for any matrix K:

$$n^{-2}\mathrm{Tr}(KGM) = n^{-4}\sum_{\mu=1}^{m}\frac{(KG^\mu X^\mu, X^\mu)}{1 + n^{-2}(G^\mu X^\mu, X^\mu)}.$$

On the next step we want to find the mathematical expectation of both sides. Since G^μ and X^μ are independent

$$E_\mu\{(KG^\mu X^\mu, X^\mu)\} = \mathrm{Tr}(KG^\mu BJB).$$

The idea of proof (2)

It's convenient to find expectation separately of numerator and denominator. For this we need to prove that for any bounded matrices K:

$$Var\{n^{-2}(KG^{\mu}X^{\mu},X^{\mu})\}=o(1), n\to+\infty.$$

This part of the proof is the most difficult (algebraically) because of the special form of our matrices.
To replace G^{μ} with G we need the next expression:

$$n^{-2}|\mathrm{Tr}K(G-G^{\mu})|=O(n^{-2})$$

which holds for any bounded matrix K.

The idea of proof (3)

At the end we need to prove that

$$Var\{n^{-2}\mathrm{Tr}KG\} \le \frac{c}{n^2}.$$

These facts we prove using standard scheme which can be used for many different ensembles.
Above relations imply that for any bounded non-random matrix K:

$$\frac{1}{n^2}E\{Tr(KGM)\} = \frac{c_n n^{-2}E\{Tr(KGBJB)\}}{1 + n^{-2}E\{Tr(GBJB)\}} + o(1),\ n \to \infty.$$

Taking

$$K = (c_n(1 + n^{-2}E\{\mathrm{Tr}(GBJB)\})^{-1}BJB - z)^{-1}$$

and $K = I$, we obtain

$$f_n(z) = f_n^{(0)}\left(\frac{z}{c_n - zf_n(z) - 1}\right)(c_n - zf_n(z) - 1)^{-1} + o(1).$$