Inverse problems and sharp eigenvalue asymptotics for Euler-Bernoulli operators

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September 12, 2016

Trilateral Ukrainian-Russian-German Summer School
Spectral Theory, Differential Equations and Probability
Mainz 2016

The Euler-Bernoulli operator

We consider the Euler-Bernoulli operator $\mathcal{E}$ given by

$$\mathcal{E}u = \frac{1}{b}(au'')'',$$

on the interval $[0,1]$ with the boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where the coefficients a, b are positive.

The operator $\mathcal{E}$ describes the relationship between the pinned-pinned beam's deflection and the applied load, a is the rigidity and b is the density of the beam.

I. High energy asymptotics and Ambarzumyan type inverse results

Second order operators

1. Recall the following result of Ambarzumyan:
Let $\lambda_0 < \lambda_1 < ...$ be the eigenvalues of the problem

$$-y'' + V(x)y = \lambda y, \qquad x \in [0,1], \qquad y'(0) = y'(1) = 0,$$

where V is a real continuous function. Then $\lambda_n = (\pi n)^2$ for all $n = 0, 1, 2, ...,$ iff $V = 0$.

Remark. In general, the spectrum of the second order operator does not determine the potential, i.e., Ambarzumyan's theorem is not valid for other boundary conditions.

2. Let $\lambda_1 < \lambda_2 < ...$ be the eigenvalues of the problem

$$-y'' + V(x)y = \lambda y, \qquad x \in [0,1], \qquad y(0) = y(1) = 0,$$

where $V \in L^1(0,1)$. Then

$$\lambda_n = (\pi n)^2 + \int_0^1 V(x)dx + o(1) \quad \text{as} \quad n \to \infty. \tag{1}$$

3. Consider the weighted second order operator

$$-\frac{1}{b}u''_{xx}, \qquad x \in [0,1], \qquad u(0) = u(1) = 0, \tag{2}$$

with positive coefficient $b, b''_{xx} \in L^1(0,1)$, satisfying the conditions $b'_x(0) = b'_x(1) = 0$ and normalized by $\int_0^1 b^{\frac{1}{2}} ds = 1$. Let $\lambda_1 < \lambda_2 < ...$ be the eigenvalues of this operator. The Liouville transformation

$$t(x) = \int_0^x b^{\frac{1}{2}}(s)ds, \qquad y(t) = b^{\frac{1}{4}}(x(t))u(x(t)),$$

yields that the operator (2) is unitarily equivalent to the operator

$$-y''_{tt} + (\beta^2 + \beta'_t)y, \qquad t \in [0,1], \qquad y(0) = y(1) = 0,$$

where $\beta = \frac{b'_x}{4b}$. The asymptotics (1) gives

$$\lambda_n = (\pi n)^2 + \int_0^1 \beta^2(t)dt + o(1) \qquad \text{as} \quad n \to \infty.$$

Assume that $\lambda_n = (\pi n)^2$ for all $n \in \mathbb{N}$. Then the last asymptotics implies $\beta = 0$, which yields $b = 1$. We obtain a result, similar to the result of Ambarzumyan: the eigenvalues $\lambda_n = (\pi n)^2$ for all $n \in \mathbb{N}$ iff $b = 1$.

Fourth order operators

Consider the self-adjoint operator

$$Hy = y'''' + 2(py')' + qy,$$

acting on $L^2(0,1)$ with the boundary conditions

$$y(0) = y''(0) = y(1) = y''(1) = 0,$$

where $p, q \in L^1(0,1)$. The spectrum of the operator H is discrete and consists of the eigenvalues of multiplicity $\leqslant 2$, indexed by

$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant ...,$$

counting with multiplicities. They satisfy

$$\lambda_n = (\pi n)^4 - 2(\pi n)^2 \int_0^1 p(t)dt + o(n^2), \quad n \to \infty. \tag{3}$$

If $p = q = 0$, then $\lambda_n = (\pi n)^4, n \in \mathbb{N}$.

The Euler-Bernoulli operator

Consider the Euler-Bernoulli operator $\mathcal{E}$ given by

$$\mathcal{E}u = \frac{1}{b}(au'')'',$$

acting on $L^2((0,1), b(x)dx)$ with the boundary conditions

$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

We assume that the coefficients a, b satisfy

$$a'(0) = a'(1) = b'(0) = b'(1) = 0, \quad a, b > 0, \quad a'', b'' \in L^1(0,1),$$

and they are normalized by the conditions $a(0) = 1, \int_0^1 (\frac{b}{a})^{\frac{1}{4}} dx = 1$.
A unitary Liouville type transformation shows that the operator $\mathcal{E}$ is unitarily equivalent to the operator H. Then the eigenvalues $\lambda_n, n \geqslant 1$ of $\mathcal{E}$ coincide with the eigenvalues of H. In particular, in the case of a uniform beam (i.e., $a, b = 1$) the corresponding eigenvalues have the form $\lambda_n = (\pi n)^4, n \in \mathbb{N}$.

The asymptotics (3) gives the following Ambarzumyan type inverse result.

Theorem (Badanin, Korotyaev, 2015)

i) The eigenvalues λ_n of the operator $\mathcal{E}$ satisfy

$$\lambda_n = (\pi n)^4 + (\pi n)^2 \psi_0 + o(n^2) \qquad \text{as} \quad n \to \infty,$$

where

$$\psi_0 = \int_0^1 \frac{5\alpha^2 + 5\beta^2 + 6\alpha\beta}{4\xi} dx \geqslant 0, \quad \alpha = \frac{a'}{4a}, \quad \beta = \frac{b'}{4b}, \quad \xi = \left(\frac{b}{a}\right)^{\frac{1}{4}}.$$

ii) $\lambda_n = (\pi n)^4$ for all $n \geqslant 1$ iff $a = b = 1$.

Remark. 1) We have $\psi_0(\alpha, \beta) = \psi_0(-\beta, -\alpha) \geqslant 0$ and $\psi_0(\alpha, \beta) = 0$ iff $\alpha = \beta = 0$. Any perturbation of the coefficients a, b moves strongly all large eigenvalues to the right. Moreover, the first two terms in the eigenvalue asymptotics for the operators $\mathcal{E}(a, b)$ and $\mathcal{E}(\frac{1}{b}, \frac{1}{a})$ coincide.

2) We believe that these results hold for larger class of boundary conditions.

II. Near constant coefficients and Barcilon type inverse results

Second order operators

Consider the weighted operator with the coefficient close to one:

$$-\frac{1}{B_\varepsilon}u'', \qquad u(0)=u(1)=0,$$

where $B_\varepsilon(x)=\dfrac{b^\varepsilon(x)}{(\int_0^1 b^{\frac{\varepsilon}{2}}(x)dx)^2}, \quad b'\in L^1(0,1), \quad \varepsilon\in\mathbb{R}, \;\; \varepsilon\to 0,$

$$\int_0^1 B_\varepsilon^{\frac12}(x)dx=1, \qquad B_\varepsilon(x)\to 1 \quad \text{as} \quad \varepsilon\to 0$$

uniformly on $[0,1]$. Let $\lambda_1(\varepsilon)<\lambda_2(\varepsilon)<\lambda_3(\varepsilon)<...$ be its eigenvalues. Each eigenvalue $\lambda_n(\varepsilon), n\geqslant 1$, satisfies

$$\lambda_n(\varepsilon)=(\pi n)^2-2\pi n\widehat{\beta}_{sn}\varepsilon+O(\varepsilon^2)$$

as $\varepsilon\to 0$, where $\beta=\dfrac{b'}{4b}, \;\; \widehat{\beta}_{sn}=\displaystyle\int_0^1\beta(t)\sin(2\pi nt)dt$. The asymptotics shows that each perturbed eigenvalue $\lambda_n(\varepsilon)$ remains close to the unperturbed one $\lambda_n(0)=(\pi n)^2$ under the small perturbations. They can move to the left or to the right.

In particular,

$$\lambda_n'(0) = -2\pi n \widehat{\beta}_{sn}.$$

Assume that for some unknown

$$\beta \in L^1_{odd}(0,1) = \{f \in L^1(0,1) : f(x) = -f(1-x), x \in (0,1)\}$$

we have the sequence $\lambda_n'(0), n \in \mathbb{N}$, of derivatives of the eigenvalues at $\varepsilon = 0$. Then β is uniquely determined by

$$\beta(x) = 2\sum_{n=1}^{\infty} \widehat{\beta}_{sn} \sin 2\pi nx = -\sum_{n=1}^{\infty} \frac{\lambda_n'(0)}{\pi n} \sin 2\pi nx, \quad x \in (0,1).$$

Euler-Bernoulli operators

Consider the Euler-Bernoulli operator with the coefficients close to one:

$$\mathcal{E}_\varepsilon u = \frac{1}{B_\varepsilon}(a^\varepsilon u'')'', \qquad u(0) = u(1) = u''(0) = u''(1) = 0,$$

where

$$B_\varepsilon(x) = \frac{b^\varepsilon(x)}{\left(\int_0^1 \left(\frac{b(x)}{a(x)}\right)^{\frac{\varepsilon}{4}} dx\right)^4}, \quad a', b' \in L^1(0,1), \quad \varepsilon \in \mathbb{R}, \quad \varepsilon \to 0,$$

$$\int_0^1 \left(\frac{B_\varepsilon(x)}{a^\varepsilon(x)}\right)^{\frac{1}{4}} dx = 1, \qquad B_\varepsilon(x) \to 1 \quad \text{as} \quad \varepsilon \to 0$$

uniformly on $[0,1]$. Let $\lambda_1(\varepsilon) \leqslant \lambda_2(\varepsilon) \leqslant \lambda_3(\varepsilon) \leqslant ...$ be the eigenvalues of $\mathcal{E}_\varepsilon$ labeled counting with multiplicity.

Theorem (Badanin, Korotyaev, 2015)

The eigenvalues $\lambda_n(\varepsilon)$, $n \geqslant 1$, *of the operator* $\mathcal{E}_\varepsilon$ *satisfy*

$$\lambda_n(\varepsilon) = (\pi n)^4 + 2(\pi n)^3(\widehat{\alpha}_{sn} - \widehat{\beta}_{sn})\varepsilon + O(\varepsilon^2)$$

as $\varepsilon \to 0$, *where*

$$\alpha = \frac{a'}{4a}, \qquad \beta = \frac{b'}{4b}, \qquad \widehat{f}_{sn} = \int_0^1 f(t)\sin(2\pi nt)dt.$$

Remark. The asymptotics shows that each perturbed eigenvalue $\lambda_n(\varepsilon)$ remains close to the unperturbed one $\lambda_n(0) = (\pi n)^4$ under the small perturbations. They can move to the left or to the right. In particular,

$$\lambda_n'(0) = 2(\pi n)^3(\widehat{\alpha}_{sn} - \widehat{\beta}_{sn}). \qquad (4)$$

If $\widehat{\alpha}_{sn} > \widehat{\beta}_{sn}$, we obtain $\lambda_n(\varepsilon) > \lambda_n(0)$. If $\widehat{\alpha}_{sn} < \widehat{\beta}_{sn}$, then we obtain $\lambda_n(\varepsilon) < \lambda_n(0)$.

Assume that $\alpha \in L^1(0,1)$ and for some unknown

$$\beta \in L^1_{odd}(0,1) = \{f \in L^1(0,1) : f(x) = f(1-x), x \in (0,1)\}$$

we have the sequence $\lambda_n'(0), n \in \mathbb{N}$, of derivatives of the eigenvalues of the operator $\mathcal{E}_\varepsilon$ at $\varepsilon = 0$. Identity (4) gives

$$\frac{\alpha(x) - \alpha(-x)}{2} - \beta(x) = 2\sum_{n=1}^{\infty}(\widehat{\alpha}_{sn} - \widehat{\beta}_{sn}) \sin 2\pi nx = \sum_{n=1}^{\infty} \frac{\lambda_n'(0)}{(\pi n)^3} \sin 2\pi nx,$$

for all $x \in (0,1)$. Then β is uniquely determined by

$$\beta(x) = \frac{\alpha(x) - \alpha(-x)}{2} - \sum_{n=1}^{\infty} \frac{\lambda_n'(0)}{(\pi n)^3} \sin 2\pi nx, \qquad x \in (0,1).$$

Remarks. 1) Similar arguments show that the function $\alpha \in L^1_{odd}(0,1)$ can be determined by $\beta \in L^1(0,1)$ and $\lambda_n'(0), n \in \mathbb{N}$.

2) Barcilon (1987) proved that the Euler-Bernoulli operator can be determined by the sequences $(\lambda_n'(0))_{n=1}^{\infty}$ for three different boundary problems.

III. Sharp high energy asymptotics

Second order operator

Let $\lambda_1 < \lambda_2 < ...$ be the eigenvalues of the problem

$$-y'' + V(x)y = \lambda y, \qquad x \in [0,1], \qquad y(0) = y(1) = 0,$$

where $V \in L^1(0,1)$. Then

$$\lambda_n = (\pi n)^2 + \int_0^1 V(x)dx - \int_0^1 V(x)\cos 2\pi nx dx + \frac{o(1)}{n} \quad \text{as} \quad n \to \infty.$$

Fourth order operator

Consider the self-adjoint operator

$$Hy = y'''' + 2(py')' + qy,$$

acting on $L^2(0,1)$ with the boundary conditions

$$y(0) = y''(0) = y(1) = y''(1) = 0,$$

Caudill-Perry-Schueller (1998) proved that in the case $p, q \in L^1(0,1)$

$$\lambda_n = (\pi n)^4 - 2(\pi n)^2 \left(\widehat{p}_0 + \widehat{p}_{cn}\right) + O(n^{1+\varepsilon}), \quad n \to \infty,$$

for any $\varepsilon > 0$ small enough,

$$\widehat{p}_{cn} = \int_0^1 p(t)\cos(2\pi nt)dt, \widehat{p}_0 = \int_0^1 p(t)dt.$$

If $p \in L^1(0,1), p' \notin L^1(0,1)$, then there are no q in the leading terms.

In order to get q in leading terms we assume $p'' \in L^1(0,1)$.

Theorem (Badanin, Korotyaev, 2015)

Let $p'', q \in L^1(0,1)$. Then the eigenvalues λ_n of the operator H satisfy

$$\lambda_n = (\pi n)^4 - 2(\pi n)^2 \widehat{p}_0 - \frac{1}{2}\int_0^1 (p(t) - \widehat{p}_0)^2 dt + \widehat{V}_0 - \widehat{V}_{cn} + \frac{o(1)}{n}$$

as $n \to \infty$, where

$$V = q - \frac{p''}{2}, \quad \widehat{V}_{cn} = \int_0^1 V(t)\cos(2\pi n t)dt, \quad \widehat{V}_0 = \int_0^1 V(t)dt.$$

Remark. The result holds for complex p, q.

The Euler-Bernoulli operator

Consider the Euler-Bernoulli operator $\mathcal{E}$ given by

$$\mathcal{E}u = \frac{1}{b}(au'')'',$$

$$u(0) = u(1) = u''(0) = u''(1) = 0,$$

where the coefficients a, b are 1-periodic, satisfy

$$a, b > 0, \quad a'''', b'''' \in L^1(\mathbb{R}),$$

and normalized by the conditions $a(0) = 1$, $\int_0^1 (\frac{b}{a})^{\frac{1}{4}} dx = 1$.

Theorem (Badanin, Korotyaev, 2015)

The eigenvalues λ_n of the operator $\mathcal{E}$ satisfy

$$\lambda_n = (\pi n)^4 + (\pi n)^2\psi_0 + \psi_1 - \gamma_{cn} + \frac{o(1)}{n} \qquad as \quad n \to \infty,$$

where $\psi_0 = \int_0^1 \frac{5\alpha^2 + 5\beta^2 + 6\alpha\beta}{2\xi} dx \geqslant 0, \quad \alpha = \frac{a'}{4a}, \quad \beta = \frac{b'}{4b},$

$$\psi_1 = \int_0^1 \left(\frac{(\sigma^2 - \varphi)^2}{2} - \left(\frac{\sigma'}{\xi} - \varphi \right)^2 \right) \xi dx + \frac{\psi_0^2}{2}, \quad \xi = \left(\frac{b}{a} \right)^{\frac{1}{4}},$$

$$\gamma_{cn} = \int_0^1 \frac{\alpha'''(x) - \beta'''(x)}{4\xi^3(x)} \cos\left(2\pi n \int_0^x \xi(s) ds \right) dx,$$

$$\sigma = \frac{\alpha + 3\beta}{2\xi}, \quad \varphi = \frac{1}{2\xi} \left(\frac{\beta - \alpha}{\xi} \right)' + \frac{\beta^2 - \alpha^2}{\xi^2}.$$

Remark. If we introduce a variable $t = \int_0^x \xi(s) ds$, then γ_{cn} become Fourier coefficients of the function $\frac{b}{4a}(\alpha''' - \beta''')$.

Thank you for attention!