Transformation operators in control problems

Larissa V. Fardigola

B. Verkin Institute for Low Temperature Physics and Engineering
of the National Academy of Sciences of Ukraine
Kharkiv, Ukraine

September 5–14, 2016

Contents

- Transformation operators

Contents

Contents

- Transformation operators
- Controllability problems for the wave equation with constant coefficients
- Controllability problems for the wave equation with variable coefficients

Transformation operators for the Sturm–Liouville operators

Let $\mathbf{T}_r : L^2(0,+\infty) \to L^2(0,+\infty)$ be the well-known transformation operator saving the asymptotics at infinity:

$$\left(\frac{d^2}{d\lambda^2} - r(\lambda)\right)\mathbf{T}_r g = \mathbf{T}_r \frac{d^2}{d\xi^2} g, \quad g \in H^2(0,+\infty),$$

Transformation operators for the Sturm–Liouville operators

Let $\mathbf{T}_r : L^2(0,+\infty) \to L^2(0,+\infty)$ be the well-known transformation operator saving the asymptotics at infinity:

$$\left(\frac{d^2}{d\lambda^2} - r(\lambda)\right)\mathbf{T}_r g = \mathbf{T}_r \frac{d^2}{d\xi^2} g, \quad g \in H^2(0,+\infty),$$

where $r \in C^1[0,+\infty) \bigcap L^\infty(0,+\infty)$ i $\int_0^\infty \lambda |r(\lambda)|\, d\lambda < \infty$. Its properties are given in the book of V.A. Marchenko, "Sturm–Liouville Operators and Applications", 2011. Various transformation operators were studied by M.Jaulent, C.Jean, E.Ya.Khruslov, B.Ya.Levin, B.M.Levitan, A.Ya.Povzner and other mathematicians.

Transformation operators for the Sturm–Liouville operators

Let $\mathbf{T}_r : L^2(0,+\infty) \to L^2(0,+\infty)$ be the well-known transformation operator saving the asymptotics at infinity:

$$\left(\frac{d^2}{d\lambda^2} - r(\lambda)\right)\mathbf{T}_r g = \mathbf{T}_r \frac{d^2}{d\xi^2} g, \quad g \in H^2(0,+\infty),$$

where $r \in C^1[0,+\infty) \bigcap L^\infty(0,+\infty)$ i $\int_0^\infty \lambda |r(\lambda)|\, d\lambda < \infty$. Its properties are given in the book of V.A. Marchenko, "Sturm–Liouville Operators and Applications", 2011. Various transformation operators were studied by M.Jaulent, C.Jean, E.Ya.Khruslov, B.Ya.Levin, B.M.Levitan, A.Ya.Povzner and other mathematicians.

This operator has been extended to the classical Sobolev spaces $H^{-m} = H^{-m}(\mathbb{R})$, $m = 0, 1, 2$ by L.V.Fardigola [SIAM J. Control Optim. **51** (2013), 1781–180], [Mathematical Control and Related Fields **5** (2015), 31–53] and by K.S.Khalina [Dopovidi Nats. Acad. Nauk. Ukr. (2012), No. 10, 24–29].

Transformation operator $\mathbf{T}_r : L^2(0, +\infty) \to L^2(0, +\infty)$

Operator $\mathbf{T}_r$ transforms each $L^2(0, +\infty)$-solution to

$$-v'' = \mu^2 v, \qquad \xi > 0, \tag{1}$$

Transformation operator $\mathbf{T}_r : L^2(0,+\infty) \to L^2(0,+\infty)$

Operator $\mathbf{T}_r$ transforms each $L^2(0,+\infty)$-solution to

$$-v'' = \mu^2 v, \qquad \xi > 0, \tag{1}$$

into an $L^2(0,+\infty)$-solution to

$$-y'' + r(\xi) = \mu^2 v, \qquad \xi > 0, \tag{2}$$

Transformation operator $\mathbf{T}_r : L^2(0,+\infty) \to L^2(0,+\infty)$

Operator $\mathbf{T}_r$ transforms each $L^2(0,+\infty)$-solution to

$$-v'' = \mu^2 v, \qquad \xi > 0, \tag{1}$$

into an $L^2(0,+\infty)$-solution to

$$-y'' + r(\xi) = \mu^2 v, \qquad \xi > 0, \tag{2}$$

under boundary condition $\frac{y(\xi,\mu)}{v(\xi,\mu)} \to 1$ as $\xi \to +\infty$, $\mu \in \mathbb{C}$. It is known that $D(\mathbf{T}_r) = R(\mathbf{T}_r) = L^2(0,+\infty)$, the operator $\mathbf{T}_r$ is bounded and invertible.

Transformation operator $\mathbf{T}_r : L^2(0,+\infty) \to L^2(0,+\infty)$

Operator $\mathbf{T}_r$ transforms each $L^2(0,+\infty)$-solution to

$$-v'' = \mu^2 v, \qquad \xi > 0, \tag{1}$$

into an $L^2(0,+\infty)$-solution to

$$-y'' + r(\xi) = \mu^2 v, \qquad \xi > 0, \tag{2}$$

under boundary condition $\frac{y(\xi,\mu)}{v(\xi,\mu)} \to 1$ as $\xi \to +\infty$, $\mu \in \mathbb{C}$. It is known that $D(\mathbf{T}_r) = R(\mathbf{T}_r) = L^2(0,+\infty)$, the operator $\mathbf{T}_r$ is bounded and invertible. Moreover

$$(\mathbf{T}_r g)(\lambda) = g(\lambda) + \int_x^\infty K(\lambda,\xi) g(\xi)\, d\xi, \qquad \lambda > 0,$$

$$\left(\mathbf{T}_r^{-1} f\right)(\xi) = f(\xi) + \int_\xi^\infty L(\xi,x) f(x)\, dx, \qquad \xi > 0,$$

Transformation operator $\mathbf{T}_r : L^2(0,+\infty) \to L^2(0,+\infty)$

Operator $\mathbf{T}_r$ transforms each $L^2(0,+\infty)$-solution to

$$-v'' = \mu^2 v, \qquad \xi > 0, \tag{1}$$

into an $L^2(0,+\infty)$-solution to

$$-y'' + r(\xi) = \mu^2 v, \qquad \xi > 0, \tag{2}$$

under boundary condition $\frac{y(\xi,\mu)}{v(\xi,\mu)} \to 1$ as $\xi \to +\infty$, $\mu \in \mathbb{C}$. It is known that $D(\mathbf{T}_r) = R(\mathbf{T}_r) = L^2(0,+\infty)$, the operator $\mathbf{T}_r$ is bounded and invertible. Moreover

$$\left(\mathbf{T}_r g\right)(\lambda) = g(\lambda) + \int_x^\infty K(\lambda,\xi) g(\xi)\, d\xi, \qquad \lambda > 0,$$

$$\left(\mathbf{T}_r^{-1} f\right)(\xi) = f(\xi) + \int_\xi^\infty L(\xi,x) f(x)\, dx, \qquad \xi > 0,$$

where K and L are well-known kernels of the transformation operator and its inverse.

The kernels K and L

The kernel K is defined by the system

$$\begin{cases} K_{y_1 y_1} - K_{y_2 y_2} = r(y_1) K, & y_2 \geq y_1 \geq 0, \\ K(y_1, y_1) = \dfrac{1}{2} \displaystyle\int_{y_1}^{\infty} r(\xi)\, d\xi, & y_1 > 0 \\ \displaystyle\lim_{y_1 + y_2 \to \infty} K_{y_1}(y) - K_{y_2}(y) = 0, & y_2 \geq y_1 \geq 0. \end{cases} \tag{3}$$

The kernels K and L

The kernel K is defined by the system

$$\begin{cases} K_{y_1 y_1} - K_{y_2 y_2} = r(y_1) K, & y_2 \geq y_1 \geq 0, \\ K(y_1, y_1) = \dfrac{1}{2} \displaystyle\int_{y_1}^{\infty} r(\xi)\, d\xi, & y_1 > 0 \\ \displaystyle\lim_{y_1 + y_2 \to \infty} K_{y_1}(y) - K_{y_2}(y) = 0, & y_2 \geq y_1 \geq 0. \end{cases} \tag{3}$$

Then $L \in C^2(\{y \in \mathbb{R}^2 \mid y_2 \geq y_1 \geq 0\})$ is determined by

$$L(y) + K(y) + \int_{y_1}^{y_2} L(y_1, \xi) K(\xi, y_2)\, d\xi = 0, \qquad y_2 \geq y_1 \geq 0, \tag{4}$$

or

$$L(y) + K(y) + \int_{y_1}^{y_2} K(y_1, \xi) L(\xi, y_2)\, d\xi = 0, \qquad y_2 \geq y_1 \geq 0. \tag{5}$$

Properties of the kernels K and L I

Lemma (V.A. Marchenko, "Sturm–Liouville Operators and Applications", 2011)

Let K be a solution to (3). Then $K \in C^2(\{y \in \mathbb{R}^2 \mid y_2 \geq y_1 \geq 0\})$ and

$$|K(y)| \leq M_0 \sigma_0 \left(\frac{y_1 + y_2}{2} \right), \qquad y_2 \geq y_1 \geq 0 \tag{6}$$

$$\left| K_{y_j}(y) \right| \leq \frac{1}{4} \left| r\left(\frac{y_1 + y_2}{2} \right) \right| + M_1 \sigma_0 \left(\frac{y_1 + y_2}{2} \right), \quad y_2 \geq y_1 \geq 0, \ j = 1, 2. \tag{7}$$

where $M_0 > 0$, $M_1 > 0$, and $\sigma_0(x) = \int_x^\infty |r(\xi)| \, d\xi$, $x > 0$.

Properties of the kernels K and L II

Lemma

Let K be a solution to (3), $L \in C^2(\{y \in \mathbb{R}^2 \mid y_2 \geq y_1 \geq 0\})$ satisfy (4) or (5). Then

$$|L(y)| \leq N_0 \sigma_0 \left(\frac{y_1 + y_2}{2}\right), \qquad y_2 \geq y_1 \geq 0 \tag{8}$$

$$\left|L_{y_j}(y)\right| \leq \frac{1}{4}\left|r\left(\frac{y_1 + y_2}{2}\right)\right| + N_1 \sigma_0 \left(\frac{y_1 + y_2}{2}\right), \quad y_2 \geq y_1 \geq 0, \ j = 1, 2, \tag{9}$$

where $N_0 > 0$ and $N_1 > 0$.

Classical Sobolev spaces

Let $p \in \mathbb{N} \cup \{0\}$. Denote

$$H^p = H^p(\mathbb{R}) = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0, p} \quad \frac{d^k}{dx^k}\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|^p = \left(\sum_{k=0}^{p} \left(\left\|\frac{d^k}{dx^k}\psi\right\|_{L^2(\mathbb{R})}\right)^2\right)^{1/2},$$

Classical Sobolev spaces

Let $p \in \mathbb{N} \cup \{0\}$. Denote

$$H^p = H^p(\mathbb{R}) = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0, p} \quad \frac{d^k}{dx^k}\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|^p = \left(\sum_{k=0}^{p} \left(\left\| \frac{d^k}{dx^k}\psi \right\|_{L^2(\mathbb{R})} \right)^2 \right)^{1/2},$$

$$H^{-p} = (H^p)^*,$$

$$\|f\|^{-p} = \sup \left\{ \frac{|\langle f, \varphi \rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0 \right\},$$

$$\left\langle \frac{d}{dx} f, \varphi \right\rangle = - \left\langle f, \frac{d}{dx}\varphi \right\rangle.$$

Classical Sobolev spaces

Let $p \in \mathbb{N} \cup \{0\}$. Denote

$$H^p = H^p(\mathbb{R}) = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0, p} \quad \frac{d^k}{dx^k}\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|^p = \left(\sum_{k=0}^{p} \left(\left\| \frac{d^k}{dx^k}\psi \right\|_{L^2(\mathbb{R})} \right)^2 \right)^{1/2},$$

$$H^{-p} = (H^p)^*,$$

$$\|f\|^{-p} = \sup \left\{ \frac{|\langle f, \varphi \rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0 \right\},$$

$$\left\langle \frac{d}{dx} f, \varphi \right\rangle = - \left\langle f, \frac{d}{dx}\varphi \right\rangle.$$

Denote by $\widetilde{H}^m$ the subspace of all odd distributions in H^m, $m \in \mathbb{Z}$.

Extension of $\mathbf{T}_r$ to $\widetilde{H}^0$

Suppose that function r is even extended.
Denote $\widetilde{\mathbf{T}}_0 : \widetilde{H}^0 \to \widetilde{H}^0$ with the domain $D\left(\widetilde{\mathbf{T}}_0\right) = \widetilde{H}^0$,

$$\left(\widetilde{\mathbf{T}}_0 g\right)(\lambda) = g(\lambda) + \operatorname{sgn}\lambda \int_{|\lambda|}^{\infty} K(|\lambda|, \xi) g(\xi)\, d\xi, \quad \lambda \in \mathbb{R},\ g \in D(\widetilde{\mathbf{T}}_0).$$

Extension of $\mathbf{T}_r$ to $\widetilde{H}^0$

Suppose that function r is even extended.
Denote $\widetilde{\mathbf{T}}_0 : \widetilde{H}^0 \to \widetilde{H}^0$ with the domain $D\left(\widetilde{\mathbf{T}}_0\right) = \widetilde{H}^0$,

$$\left(\widetilde{\mathbf{T}}_0 g\right)(\lambda) = g(\lambda) + \operatorname{sgn}\lambda \int_{|\lambda|}^{\infty} K(|\lambda|,\xi) g(\xi)\, d\xi, \quad \lambda \in \mathbb{R},\ g \in D(\widetilde{\mathbf{T}}_0).$$

The operator $\widetilde{\mathbf{T}}_0$ is invertible and $\widetilde{\mathbf{T}}_0^{-1} : \widetilde{H}^0 \to \widetilde{H}^0$, $D\left(\widetilde{\mathbf{T}}^{-1}\right) = \widetilde{H}^0$,

$$\left(\widetilde{\mathbf{T}}_0^{-1} f\right)(\xi) = f(\xi) + \operatorname{sgn}\xi \int_{|\xi|}^{\infty} L(|\xi|,\lambda) f(\lambda)\, d\lambda, \quad \xi \in \mathbb{R},\ f \in D(\widetilde{\mathbf{T}}_0^{-1}),$$

The adjoint operators for $\widetilde{\mathbf{T}}_0$ and $\widetilde{\mathbf{T}}_0^{-1}$

For the adjoint operators $\widetilde{\mathbf{T}}_0^*$ and $\left(\widetilde{\mathbf{T}}_0^{-1}\right)^* = \left(\widetilde{\mathbf{T}}_0^*\right)^{-1}$ we have $\widetilde{\mathbf{T}}_0^* : \widetilde{H}^0 \to \widetilde{H}^0$, $D\left(\widetilde{\mathbf{T}}_0^*\right) = \widetilde{H}^0 = R\left((\widetilde{\mathbf{T}}_0^*)^{-1}\right)$,

$$\left(\widetilde{\mathbf{T}}_0^*\varphi\right)(\xi) = \varphi(\xi) + \operatorname{sgn}\xi \int_0^{|\xi|} K(\lambda, |\xi|)\varphi(\lambda)\, d\lambda, \qquad \xi \in \mathbb{R},\ \varphi \in D\left(\widetilde{\mathbf{T}}_0^*\right),$$

The adjoint operators for $\widetilde{\mathbf{T}}_0$ and $\widetilde{\mathbf{T}}_0^{-1}$

For the adjoint operators $\widetilde{\mathbf{T}}_0^*$ and $\left(\widetilde{\mathbf{T}}_0^{-1}\right)^* = \left(\widetilde{\mathbf{T}}_0^*\right)^{-1}$ we have
$\widetilde{\mathbf{T}}_0^* : \widetilde{H}^0 \to \widetilde{H}^0$, $D\left(\widetilde{\mathbf{T}}_0^*\right) = \widetilde{H}^0 = R\left((\widetilde{\mathbf{T}}_0^*)^{-1}\right)$,

$$\left(\widetilde{\mathbf{T}}_0^*\varphi\right)(\xi) = \varphi(\xi) + \operatorname{sgn}\xi \int_0^{|\xi|} K(\lambda,|\xi|)\varphi(\lambda)\,d\lambda, \qquad \xi \in \mathbb{R},\ \varphi \in D\left(\widetilde{\mathbf{T}}_0^*\right),$$

and $\left(\widetilde{\mathbf{T}}_0^*\right)^{-1} : \widetilde{H}^0 \to \widetilde{H}^0$, $D\left((\widetilde{\mathbf{T}}_0^*)^{-1}\right) = \widetilde{H}^0 = R\left(\widetilde{\mathbf{T}}_0^*\right)$,

$$\left(\left(\widetilde{\mathbf{T}}_0^*\right)^{-1}\psi\right)(\lambda) = \psi(\lambda) + \operatorname{sgn}\lambda \int_0^{|\lambda|} L(\xi,|\lambda|)\psi(\xi)\,d\xi,$$
$$\lambda \in \mathbb{R},\ \psi \in D\left((\widetilde{\mathbf{T}}_0^*)^{-1}\right).$$

Properties of the operator $\widetilde{\mathbf{T}}_0^*$

Theorem

- $\widetilde{\mathbf{T}}_0^*$ *is an automorphism of* $\widetilde{H}^m$, $0 \leq m \leq 2$;

Properties of the operator $\widetilde{\mathbf{T}}_0^*$

Theorem

- $\widetilde{\mathbf{T}}_0^*$ *is an automorphism of* $\widetilde{H}^m$, $0 \le m \le 2$;
- $\left(\widetilde{\mathbf{T}}_0^* \varphi\right)''(\xi) = \left(\widetilde{\mathbf{T}}_0^* \left(\dfrac{d^2}{d\lambda^2} - r\right) \varphi\right)(\xi) + \operatorname{sgn}\xi\, K_{y_1}(0,\xi)\varphi'(0)$, $\varphi \in \widetilde{H}^2$.

Properties of the operator $\widetilde{\mathbf{T}}_0^*$

Theorem

- $\widetilde{\mathbf{T}}_0^*$ *is an automorphism of* $\widetilde{H}^m$, $0 \le m \le 2$;
- $\left(\widetilde{\mathbf{T}}_0^* \varphi\right)''(\xi) = \left(\widetilde{\mathbf{T}}_0^* \left(\frac{d^2}{d\lambda^2} - r\right) \varphi\right)(\xi) + \operatorname{sgn} \xi \, K_{y_1}(0,\xi)\varphi'(0)$, $\varphi \in \widetilde{H}^2$.

To prove the first assertion the estimates for the kernels K and L and their derivatives are used.

Properties of the operator $\widetilde{\mathbf{T}}_0^*$

Theorem

- $\widetilde{\mathbf{T}}_0^*$ *is an automorphism of* $\widetilde{H}^m$, $0 \le m \le 2$;
- $\left(\widetilde{\mathbf{T}}_0^* \varphi\right)''(\xi) = \left(\widetilde{\mathbf{T}}_0^* \left(\frac{d^2}{d\lambda^2} - r\right) \varphi\right)(\xi) + \operatorname{sgn} \xi \, K_{y_1}(0, \xi)\varphi'(0)$, $\varphi \in \widetilde{H}^2$.

To prove the first assertion the estimates for the kernels K and L and their derivatives are used.

To prove the second assertion we use equation

$$K_{y_1 y_1} - K_{y_2 y_2} = r(y_1)K, \quad y_2 \ge y_1 \ge 0,$$

Properties of the operator $\widetilde{\mathbf{T}}_0^*$

Theorem

- $\widetilde{\mathbf{T}}_0^*$ *is an automorphism of* $\widetilde{H}^m$, $0 \le m \le 2$;
- $\left(\widetilde{\mathbf{T}}_0^*\varphi\right)''(\xi) = \left(\widetilde{\mathbf{T}}_0^*\left(\frac{d^2}{d\lambda^2} - r\right)\varphi\right)(\xi) + \operatorname{sgn}\xi\, K_{y_1}(0,\xi)\varphi'(0)$, $\varphi \in \widetilde{H}^2$.

To prove the first assertion the estimates for the kernels K and L and their derivatives are used.
To prove the second assertion we use equation

$$K_{y_1y_1} - K_{y_2y_2} = r(y_1)K, \quad y_2 \ge y_1 \ge 0,$$

after $\left(\widetilde{\mathbf{T}}_0^*\varphi\right)''$ is calculated:

$$\begin{aligned}\left(\widetilde{\mathbf{T}}_0^*\varphi\right)'' &= \dots + \operatorname{sgn}\xi \int_0^{|\xi|} K_{y_2y_2}(\lambda,|\xi|)\varphi(\lambda)\,d\lambda \\ &= \dots + \operatorname{sgn}\xi \int_0^{|\xi|} K_{y_1y_1}(\lambda,|\xi|)\varphi(\lambda)\,d\lambda.\end{aligned}$$

Properties of the operator $\widetilde{\mathbf{T}}_0^*$

Theorem

- $\widetilde{\mathbf{T}}_0^*$ *is an automorphism of* $\widetilde{H}^m$, $0 \le m \le 2$;
- $\left(\widetilde{\mathbf{T}}_0^*\varphi\right)''(\xi) = \left(\widetilde{\mathbf{T}}_0^*\left(\frac{d^2}{d\lambda^2} - r\right)\varphi\right)(\xi) + \operatorname{sgn}\xi\, K_{y_1}(0,\xi)\varphi'(0)$, $\varphi \in \widetilde{H}^2$.

To prove the first assertion the estimates for the kernels K and L and their derivatives are used.

To prove the second assertion we use equation

$$K_{y_1y_1} - K_{y_2y_2} = r(y_1)K, \quad y_2 \ge y_1 \ge 0,$$

after $\left(\widetilde{\mathbf{T}}_0^*\varphi\right)''$ is calculated:

$$\left(\widetilde{\mathbf{T}}_0^*\varphi\right)'' = \cdots + \operatorname{sgn}\xi \int_0^{|\xi|} K_{y_2y_2}(\lambda,|\xi|)\varphi(\lambda)\,d\lambda$$

$$= \cdots + \operatorname{sgn}\xi \int_0^{|\xi|} K_{y_1y_1}(\lambda,|\xi|)\varphi(\lambda)\,d\lambda.$$

Then, integrating by parts, we prove the assertion.

Extension of $\mathbf{T}_r$ to $\widetilde{H}^{-2}$

Denote by $\widetilde{\mathbf{T}}_r$ the operator $\left(\widetilde{\mathbf{T}}_0^*|_{\widetilde{H}^2}\right)^*$. We have $\widetilde{\mathbf{T}}_r : \widetilde{H}^{-2} \to \widetilde{H}^{-2}$, $D\left(\widetilde{\mathbf{T}}_r\right) = \widetilde{H}^{-2}$,

$$\left\langle \widetilde{\mathbf{T}}_r g, \varphi \right\rangle = \left\langle g, \widetilde{\mathbf{T}}_0^* \varphi \right\rangle, \qquad g \in D\left(\widetilde{\mathbf{T}}_r\right) = \widetilde{H}^{-2},\ \varphi \in \widetilde{H}^2.$$

Extension of $\mathbf{T}_r$ to $\widetilde{H}^{-2}$

Denote by $\widetilde{\mathbf{T}}_r$ the operator $\left(\widetilde{\mathbf{T}}_0^*|_{\widetilde{H}^2}\right)^*$. We have $\widetilde{\mathbf{T}}_r : \widetilde{H}^{-2} \to \widetilde{H}^{-2}$, $D\left(\widetilde{\mathbf{T}}_r\right) = \widetilde{H}^{-2}$,

$$\left\langle \widetilde{\mathbf{T}}_r g, \varphi \right\rangle = \left\langle g, \widetilde{\mathbf{T}}_0^* \varphi \right\rangle, \qquad g \in D\left(\widetilde{\mathbf{T}}_r\right) = \widetilde{H}^{-2},\ \varphi \in \widetilde{H}^2.$$

Then $\widetilde{\mathbf{T}}_r^{-1} = \left(\left(\widetilde{\mathbf{T}}_0^*\right)^{-1}|_{\widetilde{H}^2}\right)^*$ and $\widetilde{\mathbf{T}}_r^{-1} : \widetilde{H}^{-2} \to \widetilde{H}^{-2}$, $D\left(\widetilde{\mathbf{T}}_r^{-1}\right) = \widetilde{H}^{-2}$,

$$\left\langle \widetilde{\mathbf{T}}_r^{-1} f, \psi \right\rangle = \left\langle g, \left(\widetilde{\mathbf{T}}_0^*\right)^{-1} \psi \right\rangle, \qquad f \in D\left(\widetilde{\mathbf{T}}_r^{-1}\right) = \widetilde{H}^{-2},\ \psi \in \widetilde{H}^2.$$

Properties of the operator $\widetilde{\mathbf{T}}_r$

Theorem

- $\widetilde{\mathbf{T}}_r$ *is an automorphism of* $\widetilde{H}^m$, $-2 \le m \le 2$;

Properties of the operator $\widetilde{\mathbf{T}}_r$

Theorem

- *$\widetilde{\mathbf{T}}_r$ is an automorphism of $\widetilde{H}^m$, $-2 \le m \le 2$;*
- $$\left(\frac{d^2}{d\lambda^2} - r\right)\left(\widetilde{\mathbf{T}}_r g\right) - 2\left(\widetilde{\mathbf{T}}_r g\right)(+0)\delta' = \widetilde{\mathbf{T}}_r\left(\frac{d^2}{d\xi^2} g - 2g(+0)\delta'\right),$$
  if $g \in \widetilde{H}^0_0$ and $g(+0)$ exists;

Properties of the operator $\widetilde{\mathbf{T}}_r$

Theorem

- $\widetilde{\mathbf{T}}_r$ *is an automorphism of* $\widetilde{H}^m$, $-2 \le m \le 2$;
- $\left(\dfrac{d^2}{d\lambda^2} - r\right)\left(\widetilde{\mathbf{T}}_r g\right) - 2\left(\widetilde{\mathbf{T}}_r g\right)(+0)\delta' = \widetilde{\mathbf{T}}_r \left(\dfrac{d^2}{d\xi^2} g - 2g(+0)\delta'\right)$, *if* $g \in \widetilde{H}_0^0$ *and* $g(+0)$ *exists;*
- $\widetilde{\mathbf{T}}_r \delta' = \delta'$

Transformation operators for differential operators with variable coefficients

Let us construct an operator **S** such that

$$\frac{1}{\rho(x)}\left(k(x)(\mathbf{S}g)'\right)' = \mathbf{S}(g'') + \text{?} \quad \text{and} \quad \mathbf{S} : H^{-2} \to \text{?},$$

where $\rho, k \in C^1(\mathbb{R})$ are positive on $\mathbb{R}$.

Transformation operators for differential operators with variable coefficients

Let us construct an operator **S** such that

$$\frac{1}{\rho(x)}\left(k(x)(\mathbf{S}g)'\right)' = \mathbf{S}(g'') + \text{ ? } \quad \text{and} \quad \mathbf{S} : H^{-2} \to \text{ ? },$$

where $\rho, k \in C^1(\mathbb{R})$ are positive on $\mathbb{R}$.

Let $\eta = (k\rho)^{1/4}$, $\eta \in C^2(\mathbb{R})$, $\theta = (k/\rho)^{1/4}$,

$\sigma(x) = \int_0^x \frac{d\mu}{\theta^2(\mu)}$, $x \in \mathbb{R}$, $\sigma(x) \to +\infty$ as $x \to +\infty$,

$\mathcal{D}_{\eta\theta} = \theta^2 \left(\frac{d}{dx} + \frac{\eta'}{\eta}\right)$.

Transformation operators for differential operators with variable coefficients

Let us construct an operator $\mathbf{S}$ such that

$$\frac{1}{\rho(x)}\left(k(x)(\mathbf{S}g)'\right)' = \mathbf{S}(g'') + \text{?} \quad \text{and} \quad \mathbf{S} : H^{-2} \to \text{?},$$

where $\rho, k \in C^1(\mathbb{R})$ are positive on $\mathbb{R}$.

Let $\eta = (k\rho)^{1/4}$, $\eta \in C^2(\mathbb{R})$, $\theta = (k/\rho)^{1/4}$,

$\sigma(x) = \int_0^x \frac{d\mu}{\theta^2(\mu)}$, $x \in \mathbb{R}$, $\sigma(x) \to +\infty$ as $x \to +\infty$,

$\mathcal{D}_{\eta\theta} = \theta^2 \left(\frac{d}{dx} + \frac{\eta'}{\eta}\right)$.

Then

$$\frac{1}{\rho}\left(kf'\right)' = \mathcal{D}^2_{\eta\theta} f - \left(\mathcal{D}_{\eta\theta}\left(\theta^2 \frac{\eta'}{\eta}\right)\right) f.$$

Observations

Let f, g, φ, $\psi \in C^2(\mathbb{R})$ be functions such that the following integrals are converging. Denote

$$\varphi = S_0\psi = \frac{\psi \circ \sigma}{\eta}, \qquad \psi = S_0^{-1}\varphi = (\eta\varphi) \circ \sigma^{-1}.$$

Observations

Let f, g, φ, $\psi \in C^2(\mathbb{R})$ be functions such that the following integrals are converging. Denote

$$\varphi = S_0\psi = \frac{\psi \circ \sigma}{\eta}, \qquad \psi = S_0^{-1}\varphi = (\eta\varphi) \circ \sigma^{-1}.$$

We have

- $$\langle g, \psi \rangle = \int_{-\infty}^{\infty} g(\lambda)\psi(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \frac{g(\sigma(x))}{\eta(x)} \frac{\psi(\sigma(x))}{\eta(x)} \frac{\eta^2(x)}{\theta^2(x)}\, dx = \langle\langle S_0 g, S_0\psi \rangle\rangle,$$

  where $\langle\langle f, \varphi \rangle\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\frac{\eta^2(x)}{\theta^2(x)}\, dx$;

Observations

Let f, g, φ, $\psi \in C^2(\mathbb{R})$ be functions such that the following integrals are converging. Denote

$$\varphi = S_0\psi = \frac{\psi \circ \sigma}{\eta}, \qquad \psi = S_0^{-1}\varphi = (\eta\varphi) \circ \sigma^{-1}.$$

We have

- $$\langle g, \psi \rangle = \int_{-\infty}^{\infty} g(\lambda)\psi(\lambda)\, d\lambda$$
  $$= \int_{-\infty}^{\infty} \frac{g(\sigma(x))}{\eta(x)} \frac{\psi(\sigma(x))}{\eta(x)} \frac{\eta^2(x)}{\theta^2(x)}\, dx = \langle\langle S_0 g, S_0\psi \rangle\rangle,$$
  where $\langle\langle f, \varphi \rangle\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\frac{\eta^2(x)}{\theta^2(x)}\, dx$;
- $$\mathcal{D}_{\eta\theta} S_0\psi = \theta^2 \left(\frac{\psi' \circ \sigma}{\eta}\sigma' - \frac{\psi \circ \sigma}{\eta^2}\eta' + \frac{\eta'}{\eta}\frac{\psi \circ \sigma}{\eta} \right) = \frac{\psi' \circ \sigma}{\eta} = S_0\left(\psi'\right);$$

Observations

Let f, g, φ, $\psi \in C^2(\mathbb{R})$ be functions such that the following integrals are converging. Denote

$$\varphi = S_0\psi = \frac{\psi \circ \sigma}{\eta}, \qquad \psi = S_0^{-1}\varphi = (\eta\varphi) \circ \sigma^{-1}.$$

We have

- $$\langle g, \psi \rangle = \int_{-\infty}^{\infty} g(\lambda)\psi(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \frac{g(\sigma(x))}{\eta(x)} \frac{\psi(\sigma(x))}{\eta(x)} \frac{\eta^2(x)}{\theta^2(x)}\, dx = \langle\langle S_0 g, S_0\psi \rangle\rangle,$$
  where $\langle\langle f, \varphi \rangle\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\frac{\eta^2(x)}{\theta^2(x)}\, dx$;
- $$\mathcal{D}_{\eta\theta} S_0\psi = \theta^2 \left(\frac{\psi' \circ \sigma}{\eta}\sigma' - \frac{\psi \circ \sigma}{\eta^2}\eta' + \frac{\eta'}{\eta}\frac{\psi \circ \sigma}{\eta} \right) = \frac{\psi' \circ \sigma}{\eta} = S_0\left(\psi'\right);$$
- $$\begin{aligned}\langle\langle \mathcal{D}_{\eta\theta} f, \varphi \rangle\rangle &= \left\langle\left\langle \mathcal{D}_{\eta\theta} S_0(S_0^{-1} f), \varphi \right\rangle\right\rangle = \left\langle\left\langle S_0\left((S_0^{-1} f)'\right), S_0(S_0^{-1}\varphi) \right\rangle\right\rangle \\ &= \langle (S_0^{-1} f)', S_0^{-1}\varphi \rangle = -\langle S_0^{-1} f, (S_0^{-1}\varphi)' \rangle \\ &= -\left\langle\left\langle S_0(S_0^{-1} f), S_0\left((S_0^{-1}\varphi)'\right) \right\rangle\right\rangle = -\langle\langle f, \mathcal{D}_{\eta\theta}\varphi \rangle\rangle.\end{aligned}$$

Spaces $\mathbb{H}^m$

Operator **S** and spaces $\mathbb{H}^m$ are introduced and investigated by L.V.Fardigola [Mathematical Control and Related Fields (MCRF) **5** (2015), 31–53].

Spaces $\mathbb{H}^m$

Operator $\mathbf{S}$ and spaces $\mathbb{H}^m$ are introduced and investigated by L.V.Fardigola [Mathematical Control and Related Fields (MCRF) **5** (2015), 31–53].

Let us introduce the modified Sobolev spaces $\mathbb{H}^m$ and compare them with the classical Sobolev spaces H^m, $m = \overline{-2, 2}$.

Spaces $\mathbb{H}^m$

Operator **S** and spaces $\mathbb{H}^m$ are introduced and investigated by L.V.Fardigola [Mathematical Control and Related Fields (MCRF) **5** (2015), 31–53].

Let us introduce the modified Sobolev spaces $\mathbb{H}^m$ and compare them with the classical Sobolev spaces H^m, $m = \overline{-2, 2}$.

Let $L^2_{\eta\theta}(\mathbb{R})$ is the space with the norm

$$\|\psi\|_{L^2_{\eta\theta}(\mathbb{R})} = \sqrt{\int_{-\infty}^{\infty} |\psi(x)|^2 \frac{\eta^2(x)}{\theta^2(x)}\, dx}, \quad \psi \in L^2_{\eta\theta}(\mathbb{R})$$

Spaces $\mathbb{H}^m$

Operator $\mathbf{S}$ and spaces $\mathbb{H}^m$ are introduced and investigated by L.V.Fardigola [Mathematical Control and Related Fields (MCRF) **5** (2015), 31–53].

Let us introduce the modified Sobolev spaces $\mathbb{H}^m$ and compare them with the classical Sobolev spaces H^m, $m = \overline{-2,2}$.

Let $L^2_{\eta\theta}(\mathbb{R})$ is the space with the norm

$$\|\psi\|_{L^2_{\eta\theta}(\mathbb{R})} = \sqrt{\int_{-\infty}^{\infty} |\psi(x)|^2 \frac{\eta^2(x)}{\theta^2(x)}\,dx}, \quad \psi \in L^2_{\eta\theta}(\mathbb{R})$$

and the inner product

$$\langle\!\langle \psi_1, \psi_2 \rangle\!\rangle = \int_{-\infty}^{\infty} \psi_1(x)\psi_2(x) \frac{\eta^2(x)}{\theta^2(x)}\,dx, \quad \psi_1, \psi_2 \in L^2_{\eta\theta}(\mathbb{R}).$$

Denote by $\langle f, \varphi \rangle$ and $\langle\!\langle g, \psi \rangle\!\rangle$ the value of distributions $f \in H_0^{-p}$ and $g \in \mathbb{H}^{-p}$, respectively, on test functions $\varphi \in H_0^p$ and $\psi \in \mathbb{H}^p$, respectively.

Spaces $\mathbb{H}^m$

Operator **S** and spaces $\mathbb{H}^m$ are introduced and investigated by L.V.Fardigola [Mathematical Control and Related Fields (MCRF) **5** (2015), 31–53].

Let us introduce the modified Sobolev spaces $\mathbb{H}^m$ and compare them with the classical Sobolev spaces H^m, $m = \overline{-2, 2}$.

Let $L^2_{\eta\theta}(\mathbb{R})$ is the space with the norm

$$\|\psi\|_{L^2_{\eta\theta}(\mathbb{R})} = \sqrt{\int_{-\infty}^{\infty} |\psi(x)|^2 \frac{\eta^2(x)}{\theta^2(x)}\, dx}, \quad \psi \in L^2_{\eta\theta}(\mathbb{R})$$

and the inner product

$$\langle\langle \psi_1, \psi_2 \rangle\rangle = \int_{-\infty}^{\infty} \psi_1(x)\psi_2(x) \frac{\eta^2(x)}{\theta^2(x)}\, dx, \quad \psi_1, \psi_2 \in L^2_{\eta\theta}(\mathbb{R}).$$

Denote by $\langle f, \varphi \rangle$ and $\langle\langle g, \psi \rangle\rangle$ the value of distributions $f \in H_0^{-p}$ and $g \in \mathbb{H}^{-p}$, respectively, on test functions $\varphi \in H_0^p$ and $\psi \in \mathbb{H}^p$, respectively.

Let $p = 0, 1, 2$.

Spaces $\mathbb{H}^m$

Classical Sobolev spaces

$$H^p = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0,p}\ \frac{d^k}{dx^k}\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|^p = \left(\sum_{k=0}^{p} \left\| \frac{d^k}{dx^k}\varphi \right\|^2_{L^2(\mathbb{R})}\right)^{1/2},$$

$$H^{-p} = (H^p)^*,$$

$$\|f\|^{-p} = \sup\left\{ \frac{|\langle f,\varphi\rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0 \right\},$$

$$\left\langle \frac{d}{dx} f, \varphi \right\rangle = -\left\langle f, \frac{d}{dx}\varphi \right\rangle, \ p \neq 2.$$

Modified Sobolev spaces

$$\mathbb{H}^p = \{\psi \in L^2_{\text{loc}}(\mathbb{R}) \mid \forall k = \overline{0,p}\ \mathcal{D}^k_{\eta\theta}\psi \in L^2_{\eta\theta}(\mathbb{R})\},$$

$$|||\psi|||^p = \left(\sum_{k=0}^{p} \left(\left\| \mathcal{D}^k_{\eta\theta}\psi \right)\right\|^2_{L^2_{\eta\theta}(\mathbb{R})}\right)^{1/2},$$

$$\mathbb{H}^{-p} = (\mathbb{H}^p)^*,$$

$$|||g|||^{-p} = \sup\left\{ \frac{|\langle\langle g,\psi\rangle\rangle|}{|||\psi|||^p} \mid |||\psi|||^p \neq 0 \right\},$$

$$\langle\langle \mathcal{D}_{\eta\theta} g, \psi \rangle\rangle = -\langle\langle g, \mathcal{D}_{\eta\theta}\psi \rangle\rangle, \ p \neq 2.$$

Operator S_0

Together with the spaces $\mathbb{H}^m$ consider the operator **S**. First, consider an auxiliary operator $\mathrm{S}_0 : H^0 \to \mathbb{H}^0$, $D(\mathrm{S}_0) = H^0$,

$$\mathrm{S}_0\psi = \frac{\psi \circ \sigma}{\eta}, \quad \psi \in D(\mathrm{S}_0)$$

where $\psi \circ \sigma$ in the composition of ψ i σ, i.e., $(\psi \circ \sigma)(x) = \psi(\sigma(x))$, $x \in \mathbb{R}$.

Operator S_0

Together with the spaces $\mathbb{H}^m$ consider the operator **S**. First, consider an auxiliary operator $\mathrm{S}_0 : H^0 \to \mathbb{H}^0$, $D(\mathrm{S}_0) = H^0$,

$$\mathrm{S}_0\psi = \frac{\psi \circ \sigma}{\eta}, \quad \psi \in D(\mathrm{S}_0)$$

where $\psi \circ \sigma$ in the composition of ψ i σ, i.e., $(\psi \circ \sigma)(x) = \psi(\sigma(x))$, $x \in \mathbb{R}$. By construction, the operator S_0 is invertible, $\mathrm{S}_0^{-1} : \mathbb{H}^0 \to H^0$, $D(\mathrm{S}_0^{-1}) = \mathbb{H}^0$,

$$\mathrm{S}_0^{-1}\varphi = (\eta\varphi) \circ \sigma^{-1}, \quad \varphi \in D(\mathrm{S}_0^{-1}).$$

Properties of the operator S_0

.

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

We have

- $\mathcal{D}_{\eta\theta}\mathrm{S}_0\psi = \mathrm{S}_0(\psi')$, $\psi \in H^1$,

Properties of the operator S_0

.

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

We have

- $\mathcal{D}_{\eta\theta}\mathrm{S}_0\psi = \mathrm{S}_0(\psi')$, $\psi \in H^1$,
- *The operator* S_0 *is an isometric isomorphism of* H^m *and* $\mathbb{H}^m$, $m = 0, 1, 2$.

Operator **S**

By using this theorem, we extend the operator S_0 to H^{-2}. Denote this extension by **S**. We have $\mathbf{S} : H^{-2} \to \mathbb{H}^{-2}$, $D(\mathbf{S}) = H^{-2}$,

$$\langle\!\langle \mathbf{S} g, \varphi \rangle\!\rangle = \langle g, \mathrm{S}_0^{-1} \varphi \rangle, \qquad g \in D(\mathbf{S}),\ \varphi \in D(\mathrm{S}_0^{-1}) \cap \mathbb{H}^2 = \mathbb{H}^2.$$

Operator $\mathbf{S}$

By using this theorem, we extend the operator S_0 to H^{-2}. Denote this extension by $\mathbf{S}$. We have $\mathbf{S} : H^{-2} \to \mathbb{H}^{-2}$, $D(\mathbf{S}) = H^{-2}$,

$$\langle\!\langle \mathbf{S}g, \varphi \rangle\!\rangle = \langle g, \mathrm{S}_0^{-1}\varphi \rangle, \qquad g \in D(\mathbf{S}),\ \varphi \in D(\mathrm{S}_0^{-1}) \cap \mathbb{H}^2 = \mathbb{H}^2.$$

Evidently, $\mathbf{S}$ is also invertible, $\mathbf{S}^{-1} : \mathbb{H}^{-2} \to H^{-2}$, $D(\mathbf{S}^{-1}) = \mathbb{H}^{-2}$,

$$\langle \mathbf{S}^{-1} f, \psi \rangle = \langle\!\langle f, \mathbf{S}\psi \rangle\!\rangle\,, \qquad f \in D(\mathbf{S}^{-1}),\ \psi \in D(\mathrm{S}_0) \cap H^2 = H^2.$$

Properties of the operator **S**

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- **S** *is an isometric isomorphism of* H^m *and* $\mathbb{H}^m$, $-2 \le m \le 2$;

Properties of the operator **S**

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- $\mathbf{S}$ *is an isometric isomorphism of* H^m *and* $\mathbb{H}^m$, $-2 \leq m \leq 2$;
- $\mathcal{D}_{\eta\theta}(\mathbf{S}g) = \mathbf{S}(g')$, $g \in H^m$, $-1 \leq m \leq 2$;

Properties of the operator **S**

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- $\mathbf{S}$ *is an isometric isomorphism of* H^m *and* $\mathbb{H}^m$, $-2 \le m \le 2$;
- $\mathcal{D}_{\eta\theta}(\mathbf{S}g) = \mathbf{S}(g')$, $g \in H^m$, $-1 \le m \le 2$;
- $\langle g, \psi \rangle = \langle\langle \mathbf{S}g, \mathbf{S}\psi \rangle\rangle$, $g \in H^{-m}$, $\psi \in H^m$, $0 \le m \le 2$;

Properties of the operator **S**

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- $\mathbf{S}$ *is an isometric isomorphism of* H^m *and* $\mathbb{H}^m$, $-2 \le m \le 2$;
- $\mathcal{D}_{\eta\theta}(\mathbf{S}g) = \mathbf{S}(g')$, $g \in H^m$, $-1 \le m \le 2$;
- $\langle g, \psi \rangle = \langle\langle \mathbf{S}g, \mathbf{S}\psi \rangle\rangle$, $g \in H^{-m}$, $\psi \in H^m$, $0 \le m \le 2$;
- $\mathbf{S}\delta = \eta(0)\delta$.

Properties of the operator **S**

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- **S** *is an isometric isomorphism of* H^m *and* $\mathbb{H}^m$, $-2 \le m \le 2$;
- $\mathcal{D}_{\eta\theta}\left(\mathbf{S}g\right) = \mathbf{S}\left(g'\right)$, $g \in H^m$, $-1 \le m \le 2$;
- $\langle g, \psi \rangle = \langle\langle \mathbf{S}g, \mathbf{S}\psi \rangle\rangle$, $g \in H^{-m}$, $\psi \in H^m$, $0 \le m \le 2$;
- $\mathbf{S}\delta = \eta(0)\delta$.

In particular,

$$\frac{1}{\rho}\left(k(\mathbf{S}g)'\right)' = \mathcal{D}^2_{\eta\theta}\mathbf{S}g - \nu\mathbf{S}g = \mathbf{S}(g'') - \nu\mathbf{S}g$$

where $\nu = \mathcal{D}_{\eta\theta}\left(\theta^2\frac{\eta'}{\eta}\right)$.

Space $\mathcal{D}$

Let $\mathcal{D}$ be the space of infinitely differentiable functions with compact supports, where

$$\varphi_n \to 0 \text{ as } n \to \infty \quad \text{iff} \begin{cases} \exists a > 0\ \forall n = \overline{1, \infty}\ \operatorname{supp} \varphi_n \in [-a, a] \\ \forall m = \overline{1, \infty}\ \varphi_n^{(m)} \rightrightarrows 0 \text{ as } n \to \infty \text{ on } \mathbb{R} \end{cases}$$

Let $\mathcal{D}'$ be the dual space with weak convergence.

Space $\mathcal{S}$

Let $\mathcal{S}$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}$, i.e.

$$\mathcal{S} = \left\{ \varphi \in C^\infty(\mathbb{R}) \mid \forall k = \overline{0, \infty}\ \forall m = \overline{0, \infty}\ \sup \left\{ \left| x^k \varphi^{(m)} \right| \mid x \in \mathbb{R} \right\} < \infty \right\}$$

where

$$\varphi_n \to 0 \text{ as } n \to \infty \quad \text{iff } \forall k = \overline{0, \infty}\ \forall m = \overline{0, \infty}\ x^k \varphi_n^{(m)} \rightrightarrows 0 \text{ as } n \to \infty \text{ on } \mathbb{R}.$$

Let $\mathcal{S}'$ be the dual space of tempered distributions (with weak convergence).

Properties of the classical Sobolev spaces H^m

Theorem (S.G. Gindikin and L.R. Volevich, "Distributions and convolution equations", 1992)

- *$H^m \subset H^n$ is a dense embedding, $-2 \le n \le m \le 2$.*

Properties of the classical Sobolev spaces H^m

Theorem (S.G. Gindikin and L.R. Volevich, "Distributions and convolution equations", 1992)

- $H^m \subset H^n$ *is a dense embedding,* $-2 \le n \le m \le 2$.
- $\mathcal{D} \subset \mathcal{S} \subset H^m$ *are dense embeddings,* $-2 \le m \le 2$.

Properties of the classical Sobolev spaces H^m

Theorem (S.G. Gindikin and L.R. Volevich, "Distributions and convolution equations", 1992)

- $H^m \subset H^n$ *is a dense embedding,* $-2 \le n \le m \le 2$.
- $\mathcal{D} \subset \mathcal{S} \subset H^m$ *are dense embeddings,* $-2 \le m \le 2$.
- $H^m \subset \mathcal{S}' \subset \mathcal{D}'$ *are dense embeddings,* $-2 \le m \le 2$.

Properties of the modified Sobolev spaces $\mathbb{H}^m$

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- $\mathbb{H}^m \subset \mathbb{H}^n$ *is a dense embedding,* $-2 \leq n \leq m \leq 2$.

Properties of the modified Sobolev spaces $\mathbb{H}^m$

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- $\mathbb{H}^m \subset \mathbb{H}^n$ *is a dense embedding,* $-2 \le n \le m \le 2$.
- $\mathcal{D} \subset \mathbb{H}^m$ *is a dense embedding,* $-2 \le m \le 2$.

Properties of the modified Sobolev spaces $\mathbb{H}^m$

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- $\mathbb{H}^m \subset \mathbb{H}^n$ *is a dense embedding,* $-2 \le n \le m \le 2$.
- $\mathcal{D} \subset \mathbb{H}^m$ *is a dense embedding,* $-2 \le m \le 2$.
- $\mathbb{H}^m \subset \mathcal{D}'$ *is a dense embedding,* $-2 \le m \le 2$.

Properties of the modified Sobolev spaces $\mathbb{H}^m$

Theorem (L.V.Fardigola, (MCRF) 5 (2015), 31–53)

- $\mathbb{H}^m \subset \mathbb{H}^n$ *is a dense embedding,* $-2 \leq n \leq m \leq 2$.
- $\mathcal{D} \subset \mathbb{H}^m$ *is a dense embedding,* $-2 \leq m \leq 2$.
- $\mathbb{H}^m \subset \mathcal{D}'$ *is a dense embedding,* $-2 \leq m \leq 2$.

It is shown by examples that relations between $\mathbb{H}^m$ and $\mathcal{S}$ depends on k and ρ.

Examples

Let $k = \rho$. Then $\eta = \sqrt{\rho}$, $\theta = 1$, $\sigma(x) = x$,

Examples

Let $k = \rho$. Then $\eta = \sqrt{\rho}$, $\theta = 1$, $\sigma(x) = x$,

$$\frac{\eta}{\theta}\varphi = \eta\varphi,$$

Examples

Let $k = \rho$. Then $\eta = \sqrt{\rho}$, $\theta = 1$, $\sigma(x) = x$,

$$\frac{\eta}{\theta}\varphi = \eta\varphi,$$

$$\frac{\eta}{\theta}\mathcal{D}_{\eta\theta}\varphi = \eta\left(\varphi' + \frac{\eta'}{\eta}\varphi\right) = \eta\varphi' + \eta'\varphi = (\eta\varphi)' \qquad ,$$

Examples

Let $k = \rho$. Then $\eta = \sqrt{\rho}$, $\theta = 1$, $\sigma(x) = x$,

$$\frac{\eta}{\theta}\varphi = \eta\varphi,$$

$$\frac{\eta}{\theta}\mathcal{D}_{\eta\theta}\varphi = \eta\left(\varphi' + \frac{\eta'}{\eta}\varphi\right) = \eta\varphi' + \eta'\varphi = (\eta\varphi') \qquad ,$$

$$\frac{\eta}{\theta}\mathcal{D}^2_{\eta\theta}\varphi = \eta\mathcal{D}_{\eta\theta}\left(\frac{1}{\eta}(\eta\varphi)'\right) = \eta\left(\left(\frac{1}{\eta}(\eta\varphi)'\right)' + \frac{\eta'}{\eta}\frac{1}{\eta}(\eta\varphi)'\right)$$
$$= \eta\left(\frac{1}{\eta}(\eta\varphi)'' - \frac{\eta'}{\eta^2}(\eta\varphi)' + \frac{\eta'}{\eta}\frac{1}{\eta}(\eta\varphi)'\right) = (\eta\varphi)''.$$

Examples

Let $k=\rho$. Then $\eta=\sqrt{\rho}$, $\theta=1$, $\sigma(x)=x$,

$$\frac{\eta}{\theta}\varphi=\eta\varphi,$$

$$\frac{\eta}{\theta}\mathcal{D}_{\eta\theta}\varphi=\eta\left(\varphi'+\frac{\eta'}{\eta}\varphi\right)=\eta\varphi'+\eta'\varphi=(\eta\varphi)' \qquad ,$$

$$\frac{\eta}{\theta}\mathcal{D}^2_{\eta\theta}\varphi=\eta\mathcal{D}_{\eta\theta}\left(\frac{1}{\eta}(\eta\varphi)'\right)=\eta\left(\left(\frac{1}{\eta}(\eta\varphi)'\right)'+\frac{\eta'}{\eta}\frac{1}{\eta}(\eta\varphi)'\right)$$
$$=\eta\left(\frac{1}{\eta}(\eta\varphi)''-\frac{\eta'}{\eta^2}(\eta\varphi)'+\frac{\eta'}{\eta}\frac{1}{\eta}(\eta\varphi)'\right)=(\eta\varphi)''.$$

Therefore

$$f\in\mathbb{H}^m\Leftrightarrow\sqrt{\rho}\varphi\in H^m,\qquad m=\overline{-2,2}.$$

$\mathbb{H}^p=\{\psi\in L^2_{\mathrm{loc}}(\mathbb{R})\mid\forall k=\overline{0,p}\ \mathcal{D}^k_{\eta\theta}\psi\in L^2_{\eta\theta}(\mathbb{R})\}$,
$\mathbb{H}^{-p}=(\mathbb{H}^p)^*$, $p=0,1,2$.

Examples

Thus, the following assertions hold

- Let $\rho(x) = \cosh x$, $x \in \mathbb{R}$. Then $f \in \mathbb{H}^m$ iff $\sqrt{\cosh x} f \in H^m$, $m = \overline{-2, 2}$. Therefore, $\mathcal{S} \not\subset \mathbb{H}^2$ and $\mathbb{H}^{-2} \not\subset \mathcal{S}'$.

Examples

Thus, the following assertions hold

- Let $\rho(x) = \cosh x$, $x \in \mathbb{R}$. Then $f \in \mathbb{H}^m$ iff $\sqrt{\cosh x} f \in H^m$, $m = \overline{-2, 2}$. Therefore, $\mathcal{S} \not\subset \mathbb{H}^2$ and $\mathbb{H}^{-2} \not\subset \mathcal{S}'$.
- Let $\rho(x) = 1/\cosh x$, $x \in \mathbb{R}$. Then, $f \in \mathbb{H}^m$ iff $f/\sqrt{\cosh x} \in H^m$, $m = \overline{-2, 2}$. Therefore, $\mathcal{S} \subset \mathbb{H}^2$ i $\mathbb{H}^{-2} \subset \mathcal{S}'$.

Examples

Thus, the following assertions hold

- Let $\rho(x) = \cosh x$, $x \in \mathbb{R}$. Then $f \in \mathbb{H}^m$ iff $\sqrt{\cosh x} f \in H^m$, $m = \overline{-2, 2}$. Therefore, $\mathcal{S} \not\subset \mathbb{H}^2$ and $\mathbb{H}^{-2} \not\subset \mathcal{S}'$.
- Let $\rho(x) = 1/\cosh x$, $x \in \mathbb{R}$. Then, $f \in \mathbb{H}^m$ iff $f/\sqrt{\cosh x} \in H^m$, $m = \overline{-2, 2}$. Therefore, $\mathcal{S} \subset \mathbb{H}^2$ i $\mathbb{H}^{-2} \subset \mathcal{S}'$.
- Let $\alpha \in \mathbb{R}$, $\rho(x) = (1 + x^2)^{\frac{\alpha}{2}}$, $x \in \mathbb{R}$. Then, $f \in \mathbb{H}^m$ iff $(1 + x^2)^{\frac{\alpha}{2}} f \in H^m$, i.e., $f \in H^m_\alpha$, $m = \overline{-2, 2}$. Therefore, $\mathcal{S} \subset H^2_\alpha \subset \mathbb{H}^2 \subset \mathbb{H}^{-2} \subset H^{-2}_\alpha \subset \mathcal{S}'$.

Examples

Let $\alpha \in \mathbb{R}$, $k(x) = (1 + x^2)^{\frac{\alpha+1}{2}}$, $\rho(x) = (1 + x^2)^{\frac{\alpha-1}{2}}$, $x \in \mathbb{R}$. Then, $\eta(x) = (1 + x^2)^{\frac{\alpha}{4}}$, $\theta(x) = (1 + x^2)^{\frac{1}{4}}$, $\sigma(x) = \ln\left(x + \sqrt{1 + x^2}\right)$, $x \in \mathbb{R}$.

Examples

Let $\alpha \in \mathbb{R}$, $k(x) = (1+x^2)^{\frac{\alpha+1}{2}}$, $\rho(x) = (1+x^2)^{\frac{\alpha-1}{2}}$, $x \in \mathbb{R}$. Then, $\eta(x) = (1+x^2)^{\frac{\alpha}{4}}$, $\theta(x) = (1+x^2)^{\frac{1}{4}}$, $\sigma(x) = \ln\left(x+\sqrt{1+x^2}\right)$, $x \in \mathbb{R}$. We have

$$\begin{aligned}
\frac{\eta}{\theta}\varphi =& (1+x^2)^{\frac{\alpha-1}{4}}\varphi, \\
\frac{\eta}{\theta}\mathcal{D}_{\eta\theta}\varphi =& \theta(\eta\varphi)' = \frac{\alpha}{2}x(1+x^2)^{\frac{\alpha-3}{4}}\varphi + (1+x^2)^{\frac{\alpha+1}{4}}\varphi', \\
\frac{\eta}{\theta}\mathcal{D}^2_{\eta\theta}\varphi =& \frac{\eta}{\theta}\mathcal{D}_{\eta\theta}\left(\frac{\theta^2}{\eta}(\eta\varphi)'\right) = \theta\left(\theta^2(\eta\varphi)'\right) = \frac{\alpha}{2}\left(1+\frac{\alpha}{2}x^2\right)(1+x^2)^{\frac{\alpha-5}{4}}\varphi \\
& + (\alpha+1)x(1+x^2)^{\frac{\alpha-1}{4}}\varphi' + (1+x^2)^{\frac{\alpha+3}{4}}\varphi''
\end{aligned}$$

Examples

Since

$$\mathbb{H}^p = \{\psi \in L^2_{\text{loc}}(\mathbb{R}) \mid \forall k = \overline{0, p}\ \mathcal{D}^k_{\eta\theta}\psi \in L^2_{\eta\theta}(\mathbb{R})\},$$
$$\mathbb{H}^{-p} = (\mathbb{H}^p)^*,\ p = 0, 1, 2.$$

Examples

Since

$$\mathbb{H}^p = \{\psi \in L^2_{\text{loc}}(\mathbb{R}) \mid \forall k = \overline{0,p}\ \mathcal{D}^k_{\eta\theta}\psi \in L^2_{\eta\theta}(\mathbb{R})\},$$
$$\mathbb{H}^{-p} = (\mathbb{H}^p)^*,\ p = 0, 1, 2.$$

we have

$$\varphi \in \mathbb{H}^0 \Leftrightarrow (1+x^2)^{\frac{\alpha-1}{4}}\varphi \in H_0;$$
$$\varphi \in \mathbb{H}^1 \Leftrightarrow (1+x^2)^{\frac{\alpha-1}{4}}\varphi \in H_0 \text{ and } (1+x^2)^{\frac{\alpha+1}{4}}\varphi' \in H_0;$$
$$\varphi \in \mathbb{H}^2 \Leftrightarrow (1+x^2)^{\frac{\alpha-1}{4}}\varphi \in H_0 \text{ and } (1+x^2)^{\frac{\alpha+1}{4}}\varphi' \in H_0$$
$$\text{and } (1+x^2)^{\frac{\alpha+3}{4}}\varphi'' \in H_0.$$

Operator $\widetilde{\mathbb{T}}$

Consider the operator $\widetilde{\mathbb{T}} : \widetilde{H}^{-2} \to \widetilde{\mathbb{H}}^{-2}$, $D(\widetilde{\mathbb{T}}) = \widetilde{H}^{-2}$, $\widetilde{\mathbb{T}} = \mathbf{S}\widetilde{\mathbf{T}}_r$.

Operator $\widetilde{\mathbb{T}}$

Consider the operator $\widetilde{\mathbb{T}} : \widetilde{H}^{-2} \to \widetilde{\mathbb{H}}^{-2}$, $D(\widetilde{\mathbb{T}}) = \widetilde{H}^{-2}$, $\widetilde{\mathbb{T}} = \mathbf{S}\widetilde{\mathbf{T}}_r$.

Theorem

- $\widetilde{\mathbb{T}}$ *is an isomorphism of* $\widetilde{H}^m$ *and* $\widetilde{\mathbb{H}}^m$, $-2 \leq m \leq 2$;

Operator $\widetilde{\mathbb{T}}$

Consider the operator $\widetilde{\mathbb{T}} : \widetilde{H}^{-2} \to \widetilde{\mathbb{H}}^{-2}$, $D(\widetilde{\mathbb{T}}) = \widetilde{H}^{-2}$, $\widetilde{\mathbb{T}} = \mathbf{S}\widetilde{\mathbf{T}}_r$.

Theorem

- $\widetilde{\mathbb{T}}$ *is an isomorphism of* $\widetilde{H}^m$ *and* $\widetilde{\mathbb{H}}^m$, $-2 \leq m \leq 2$;
- $\left(\mathcal{D}^2_{\eta\theta} - r \circ \sigma\right) \widetilde{\mathbb{T}} g - 2\eta^2(0)(\widetilde{\mathbb{T}} g)(+0)\mathcal{D}_{\eta\theta}\delta = \widetilde{\mathbb{T}} \left(\dfrac{d^2}{d\xi^2} g - 2g(+0)\delta' \right)$, *if* $g \in \widetilde{H}^0$ *and* $g(+0)$ *exists;*

Operator $\widetilde{\mathbb{T}}$

Consider the operator $\widetilde{\mathbb{T}} : \widetilde{H}^{-2} \to \widetilde{\mathbb{H}}^{-2}$, $D(\widetilde{\mathbb{T}}) = \widetilde{H}^{-2}$, $\widetilde{\mathbb{T}} = \mathbf{S}\widetilde{\mathbf{T}}_r$.

Theorem

- *$\widetilde{\mathbb{T}}$ is an isomorphism of $\widetilde{H}^m$ and $\widetilde{\mathbb{H}}^m$, $-2 \le m \le 2$;*
- $\left(\mathcal{D}^2_{\eta\theta} - r \circ \sigma\right) \widetilde{\mathbb{T}} g - 2\eta^2(0)(\widetilde{\mathbb{T}} g)(+0)\mathcal{D}_{\eta\theta}\delta = \widetilde{\mathbb{T}} \left(\frac{d^2}{d\xi^2} g - 2g(+0)\delta' \right)$,

  if $g \in \widetilde{H}^0$ and $g(+0)$ exists;
- $\widetilde{\mathbb{T}}\delta' = \eta(0)\mathcal{D}_{\eta\theta}\delta$.

$-w'' = \mu^2 w, \quad L^2(0,+\infty)$-solutions

$\mathbf{T}_r$

$\mathbb{T} = \mathbf{S}\mathbf{T}_r$

$-y'' + ry = \mu^2 y, \quad L^2(0,+\infty)$-solutions

$\mathbf{S}$

$-\frac{1}{\rho}(kz')' = \mu^2 y, \quad L^2_{\eta\theta}(0,+\infty)$-solutions

$$\mu \in \mathbb{C},\ r = \left(\mathcal{D}_{\eta\theta}\left(\theta^2 \frac{\eta'}{\eta}\right)\right) \circ \sigma^{-1},\ \eta = (k\rho)^{1/4},\ \theta = (k/\rho)^{1/4},$$

$$\sigma(x) = \int_0^x \frac{d\mu}{\theta^2(\mu)},\ \mathcal{D}_{\eta\theta} = \theta^2\left(\frac{d}{dx} + \frac{\eta'}{\eta}\right),$$

Linear control systems

$$\frac{d\mathbf{w}}{dt} = A\mathbf{w} + Bu, \qquad t \in (0, T), \tag{10}$$

where $T > 0$, $\mathbf{w} : [0, T] \to \mathcal{H}$ is a state of system, $u : (0, T) \to H$ is a control, $\mathcal{H}$, H are Banach spaces, $A : \mathcal{H} \to \mathcal{H}$, $B : H \to \mathcal{H}$ are linear operators.

Null-controllability problems for the wave equation

Null-controllability problems for the wave equation on domains *bounded w.r.t. space variable:*

C. Castro, L.V.Fardigola, H.O.Fattorini, M.Gugat, V.A.Ilin, W.Krabs, G.Leugering, K.S.Khalina, V.I.Korobov, J.-L.Lions, E.I.Moiseev, Y. Liu, J. Sokolowski, D.L. Russel, G.M.Sklyar, J.Vancostenoble, X.Zhang, E.Zuazua, and many others.

Null-controllability problems for the wave equation

Null-controllability problems for the wave equation on domains *bounded w.r.t. space variable:*

C. Castro, L.V.Fardigola, H.O.Fattorini, M.Gugat, V.A.Ilin, W.Krabs, G.Leugering, K.S.Khalina, V.I.Korobov, J.-L.Lions, E.I.Moiseev, Y. Liu, J. Sokolowski, D.L. Russel, G.M.Sklyar, J.Vancostenoble, X.Zhang, E.Zuazua, and many others.

Null-controllability problems for the wave equation on domains *unbounded w.r.t. space variable:*

A.Avetisyan, M.I.Belishev, L.V.Fardigola, K.S.Khalina, A.Khurshudyan, G.M.Sklyar, A.F.Vakulenko.

Classical Sobolev spaces

Let $p \in \mathbb{N} \cup \{0\}$, Ω be a domain in $\mathbb{R}$. Denote

$$H^p(\Omega) = \{\varphi \in L^2(\Omega) \mid \forall k = \overline{0, p} \ \frac{d^k}{dx^k}\varphi \in L^2(\Omega)\},$$

$$\|\varphi\|_\Omega^p = \left(\sum_{k=0}^{p} \left(\left\| \frac{d^k}{dx^k}\psi \right\|_{L^2(\Omega)} \right)^2 \right)^{1/2},$$

Classical Sobolev spaces

Let $p \in \mathbb{N} \cup \{0\}$, Ω be a domain in $\mathbb{R}$. Denote

$$H^p(\Omega) = \{\varphi \in L^2(\Omega) \mid \forall k = \overline{0,p} \ \frac{d^k}{dx^k}\varphi \in L^2(\Omega)\},$$

$$\|\varphi\|_\Omega^p = \left(\sum_{k=0}^{p}\left(\left\|\frac{d^k}{dx^k}\psi\right\|_{L^2(\Omega)}\right)^2\right)^{1/2},$$

$$H^{-p}(\Omega) = (H^p(\Omega))^*,$$

$$\|f\|_\Omega^{-p} = \sup\left\{\frac{|\langle f,\varphi\rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0\right\},$$

Classical Sobolev spaces

Let $p \in \mathbb{N} \cup \{0\}$, Ω be a domain in $\mathbb{R}$. Denote

$$H^p(\Omega) = \{\varphi \in L^2(\Omega) \mid \forall k = \overline{0, p}\ \frac{d^k}{dx^k}\varphi \in L^2(\Omega)\},$$

$$\|\varphi\|_\Omega^p = \left(\sum_{k=0}^{p} \left(\left\|\frac{d^k}{dx^k}\psi\right\|_{L^2(\Omega)}\right)^2\right)^{1/2},$$

$$H^{-p}(\Omega) = (H^p(\Omega))^*,$$

$$\|f\|_\Omega^{-p} = \sup\left\{\frac{|\langle f, \varphi\rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0\right\},$$

$\left\langle \frac{d}{dx} f, \varphi \right\rangle_\Omega$ is the value of the distribution $f \in H_\Omega^{-p}$ on the test function $\varphi \in H_\Omega^p$.

Classical Sobolev spaces

Denote $H^m = H^m(\mathbb{R})$, $\|\cdot\|^m = \|\cdot\|_{\mathbb{R}}^m$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{R}}$.

Classical Sobolev spaces

Denote $H^m = H^m(\mathbb{R})$, $\|\cdot\|^m = \|\cdot\|^m_{\mathbb{R}}$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathbb{R}}$.
We have

$$\left\langle \frac{d}{dx} f, \varphi \right\rangle = -\left\langle f, \frac{d}{dx}\varphi \right\rangle .$$

Classical Sobolev spaces

Denote $H^m = H^m(\mathbb{R})$, $\|\cdot\|^m = \|\cdot\|^m_{\mathbb{R}}$, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}}$.
We have

$$\left\langle \frac{d}{dx} f, \varphi \right\rangle = - \left\langle f, \frac{d}{dx} \varphi \right\rangle .$$

Denote by $\widetilde{H}^m$ the subspace of all odd distributions in H^m, $m \in \mathbb{Z}$.

Spaces H^m and H_m

Let $p \in \mathbb{N} \cup \{0\}$.

Sobolev spaces H^m

$$H^p = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0,p}\ \frac{d^k}{dx^k}\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|^p = \left(\sum_{k=0}^{p} \left\|\frac{d^k}{dx^k}\varphi\right\|^2\right)^{1/2},$$

$$H^{-p} = (H^p)^*,$$

$$\|f\|^{-p} = \sup\left\{\frac{|\langle f,\varphi\rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0\right\},$$

$$H^m \subset H^l,\ m \geq l.$$

Sobolev spaces H_m

$$H_p = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0,p}\ x^k\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|_p = \left(\sum_{k=0}^{p} \left\|x^k\varphi\right\|^2\right)^{1/2},$$

$$H_{-p} = (H_p)^*,$$

$$\|f\|^{-p} = \sup\left\{\frac{|\langle f,\varphi\rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0\right\},$$

$$H_m \subset H_l,\ m \geq l.$$

Spaces H^m and H_m

Let $p \in \mathbb{N} \cup \{0\}$.

Sobolev spaces H^m

$$H^p = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0,p}\ \frac{d^k}{dx^k}\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|^p = \left(\sum_{k=0}^{p} \left\|\frac{d^k}{dx^k}\varphi\right\|^2\right)^{1/2},$$

$$H^{-p} = (H^p)^*,$$

$$\|f\|^{-p} = \sup\left\{\frac{|\langle f, \varphi\rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0\right\},$$

$$H^m \subset H^l,\ m \geq l.$$

Sobolev spaces H_m

$$H_p = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0,p}\ x^k\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|_p = \left(\sum_{k=0}^{p} \left\|x^k\varphi\right\|^2\right)^{1/2},$$

$$H_{-p} = (H_p)^*,$$

$$\|f\|^{-p} = \sup\left\{\frac{|\langle f, \varphi\rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0\right\},$$

$$H_m \subset H_l,\ m \geq l.$$

Denote by $\widetilde{H}_m$ the subspace of all odd distributions in H_m, $m \in \mathbb{Z}$.

Fourier transform of H^m and H_m

$$\left(\mathcal{F}_{x\to\sigma}\varphi\right)(\sigma) = \int_{-\infty}^{\infty} e^{-i\sigma x}\varphi(x)\,dx, \qquad \varphi \in H^0 = H_0 = L^2(\mathbb{R}),$$

Fourier transform of H^m and H_m

$$\left(\mathcal{F}_{x\to\sigma}\varphi\right)(\sigma) = \int_{-\infty}^{\infty} e^{-i\sigma x}\varphi(x)\,dx, \qquad \varphi \in H^0 = H_0 = L^2(\mathbb{R}),$$

$$\left(\mathcal{F}^{-1}_{\sigma\to x}\psi\right)(x) = \int_{-\infty}^{\infty} e^{i\sigma x}\psi(\sigma)\,dx, \qquad \psi \in H^0 = H_0 = L^2(\mathbb{R}),$$

Fourier transform of H^m and H_m

$$\left(\mathcal{F}_{x\to\sigma}\varphi\right)(\sigma) = \int_{-\infty}^{\infty} e^{-i\sigma x}\varphi(x)\,dx, \qquad \varphi \in H^0 = H_0 = L^2(\mathbb{R}),$$

$$\left(\mathcal{F}^{-1}_{\sigma\to x}\psi\right)(x) = \int_{-\infty}^{\infty} e^{i\sigma x}\psi(\sigma)\,dx, \qquad \psi \in H^0 = H_0 = L^2(\mathbb{R}),$$

$\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}^{-1}\varphi\rangle$,

$(f \in H^{-p}$ and $\varphi \in H^p)$ or $(f \in H_{-p}$ and $\varphi \in H_p)$, $p \in \mathbb{N} \cup \{0\}$.

Fourier transform of H^m and H_m

$$(\mathcal{F}_{x\to\sigma}\varphi)(\sigma) = \int_{-\infty}^{\infty} e^{-i\sigma x}\varphi(x)\,dx, \qquad \varphi \in H^0 = H_0 = L^2(\mathbb{R}),$$

$$\left(\mathcal{F}^{-1}_{\sigma\to x}\psi\right)(x) = \int_{-\infty}^{\infty} e^{i\sigma x}\psi(\sigma)\,dx, \qquad \psi \in H^0 = H_0 = L^2(\mathbb{R}),$$

$$\langle \mathcal{F}f, \varphi\rangle = \langle f, \mathcal{F}^{-1}\varphi\rangle,$$
$$(f \in H^{-p} \text{ and } \varphi \in H^p) \text{ or } (f \in H_{-p} \text{ and } \varphi \in H_p),\ \ p \in \mathbb{N}\cup\{0\}.$$

Theorem

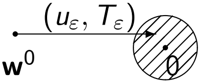
For each $m \in \mathbb{Z}$ the operator $\mathcal{F}$ is an isometric isomorphism of H^m and H_m.

Null-controllability problems

Let $\mathfrak{U}$ be a set of permissible controls.

Definition

A state $\mathbf{w}^0$ is called **approximately null-controllable at a free time** if $\forall \varepsilon > 0$ there exist $T_\varepsilon > 0$ $u_\varepsilon \in \mathfrak{U}$ such that a solution $\mathbf{w}$ of system (14) satisfies two conditions:
$\mathbf{w}(0) = \mathbf{w}^0$ and $\|\mathbf{w}(T)\| < \varepsilon$.



Null-controllability problems for the wave equation with constant coefficients

We consider the following controllability problem

$$w_{tt} = w_{xx} - q^2 w, \; x > 0, \; t \in (0, T), \tag{11}$$

Null-controllability problems for the wave equation with constant coefficients

We consider the following controllability problem

$$w_{tt} = w_{xx} - q^2 w, \ \ x > 0, \ \ t \in (0, T), \tag{11}$$

$$w(0, t) = u(t), \quad t \in (0, T), \tag{12}$$

Null-controllability problems for the wave equation with constant coefficients

We consider the following controllability problem

$$w_{tt} = w_{xx} - q^2 w, \;\; x > 0, \;\; t \in (0, T), \tag{11}$$

$$w(0,t) = u(t), \quad t \in (0, T), \tag{12}$$

$$\left.\begin{array}{r} w(x,0) = w_0^0(x) \\ w_t(x,0) = w_1^0(x) \end{array}\right\} \longrightarrow \left\{\begin{array}{r} w(x,T) = w_0^T(x) \\ w_t(x,T) = w_1^T(x) \end{array}\right. \tag{13}$$

Null-controllability problems for the wave equation with constant coefficients

We consider the following controllability problem

$$w_{tt} = w_{xx} - q^2 w, \;\; x > 0, \;\; t \in (0, T), \tag{11}$$

$$w(0, t) = u(t), \quad t \in (0, T), \tag{12}$$

$$\left.\begin{aligned} w(x,0) &= w_0^0(x) \\ w_t(x,0) &= w_1^0(x) \end{aligned}\right\} \longrightarrow \left\{\begin{aligned} w(x,T) &= w_0^T(x) \\ w_t(x,T) &= w_1^T(x) \end{aligned}\right. \tag{13}$$

where $T > 0$, $q \geq 0$, $w : [0, T] \to H^0(0, +\infty)$, $w^0 = \begin{pmatrix} w_0^0 \\ w_0^T \end{pmatrix} \in H^0(0, +\infty) \times H^{-1}(0, +\infty)$, $w^T = \begin{pmatrix} w_1^T \\ w_1^T \end{pmatrix} \in H^0(0, +\infty) \times H^{-1}(0, +\infty)$.

We also assume that $u \in \mathfrak{U} = L^\infty(0, T)$ is a control.

Reduced control problem

Let $\mathbf{w}(\cdot, t)$, $\mathbf{w}^0$, $\mathbf{w}^T$ be the odd extension for $\begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix}$, $\begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix}$, $\begin{pmatrix} w_0^T \\ w_1^T \end{pmatrix}$, resp., $(t \in [0, T])$. Then $\frac{d^p}{dt^p}\mathbf{w} : [0, T] \to \mathbf{H}^{-p}$, $p = 0, 1$, where $\mathbf{H}^m = \widetilde{H}^m \times \widetilde{H}^{m-1}$ with the norm $|||\cdot|||^m$, $m \in \mathbb{Z}$.

Reduced control problem

Let $\mathbf{w}(\cdot, t)$, $\mathbf{w}^0$, $\mathbf{w}^T$ be the odd extension for $\begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix}$, $\begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix}$, $\begin{pmatrix} w_0^T \\ w_1^T \end{pmatrix}$, resp., ($t \in [0, T]$). Then $\frac{d^p}{dt^p}\mathbf{w} : [0, T] \to \mathbf{H}^{-p}$, $p = 0, 1$, where $\mathbf{H}^m = \widetilde{H}^m \times \widetilde{H}^{m-1}$ with the norm $|||\cdot|||^m$, $m \in \mathbb{Z}$.
Our controllability problem can be reduced to the following one

$$\frac{d\mathbf{w}}{dt} = \begin{pmatrix} 0 & 1 \\ \left(\frac{d}{dx}\right)^2 - q^2 & 0 \end{pmatrix} \mathbf{w} - \begin{pmatrix} 0 \\ 2\delta'(x) \end{pmatrix} u, \quad x \in \mathbb{R}, t \in (0, T), \tag{14}$$

Reduced control problem

Let $\mathbf{w}(\cdot,t)$, $\mathbf{w}^0$, $\mathbf{w}^T$ be the odd extension for $\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}$, $\begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix}$, $\begin{pmatrix} w_0^T \\ w_1^T \end{pmatrix}$, resp., ($t \in [0,T]$). Then $\frac{d^p}{dt^p}\mathbf{w} : [0,T] \to \mathbf{H}^{-p}$, $p = 0, 1$, where $\mathbf{H}^m = \widetilde{H}^m \times \widetilde{H}^{m-1}$ with the norm $|||\cdot|||^m$, $m \in \mathbb{Z}$.
Our controllability problem can be reduced to the following one

$$\frac{d\mathbf{w}}{dt} = \begin{pmatrix} 0 & 1 \\ \left(\frac{d}{dx}\right)^2 - q^2 & 0 \end{pmatrix} \mathbf{w} - \begin{pmatrix} 0 \\ 2\delta'(x) \end{pmatrix} u, \quad x \in \mathbb{R}, t \in (0,T), \tag{14}$$

$$\mathbf{w}(\cdot,0) = \mathbf{w}^0 \to \mathbf{w}(\cdot,T) = \mathbf{w}^T, \tag{15}$$

Reduced control problem

Let $\mathbf{w}(\cdot,t)$, $\mathbf{w}^0$, $\mathbf{w}^T$ be the odd extension for $\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}$, $\begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix}$, $\begin{pmatrix} w_0^T \\ w_1^T \end{pmatrix}$, resp., ($t \in [0,T]$). Then $\frac{d^p}{dt^p}\mathbf{w} : [0,T] \to \mathbf{H}^{-p}$, $p = 0, 1$, where $\mathbf{H}^m = \widetilde{H}^m \times \widetilde{H}^{m-1}$ with the norm $|\!|\!|\cdot|\!|\!|^m$, $m \in \mathbb{Z}$.
Our controllability problem can be reduced to the following one

$$\frac{d\mathbf{w}}{dt} = \begin{pmatrix} 0 & 1 \\ \left(\frac{d}{dx}\right)^2 - q^2 & 0 \end{pmatrix} \mathbf{w} - \begin{pmatrix} 0 \\ 2\delta'(x) \end{pmatrix} u, \quad x \in \mathbb{R}, t \in (0,T), \tag{14}$$

$$\mathbf{w}(\cdot,0) = \mathbf{w}^0 \to \mathbf{w}(\cdot,T) = \mathbf{w}^T, \tag{15}$$

where δ is the Dirac distribution, $\delta = H'$, H is the Heaviside function: $H(\xi) = 1$ if $\xi > 0$, and $H(\xi) = 0$ otherwise.

Reduced control problem

Let $\mathbf{w}(\cdot, t)$, $\mathbf{w}^0$, $\mathbf{w}^T$ be the odd extension for $\begin{pmatrix} w(\cdot,t) \\ w_t(\cdot,t) \end{pmatrix}$, $\begin{pmatrix} w_0^0 \\ w_1^0 \end{pmatrix}$, $\begin{pmatrix} w_0^T \\ w_1^T \end{pmatrix}$, resp., ($t \in [0, T]$). Then $\frac{d^p}{dt^p}\mathbf{w} : [0, T] \to \mathbf{H}^{-p}$, $p = 0, 1$, where $\mathbf{H}^m = \widetilde{H}^m \times \widetilde{H}^{m-1}$ with the norm $|||\cdot|||^m$, $m \in \mathbb{Z}$.
Our controllability problem can be reduced to the following one

$$\frac{d\mathbf{w}}{dt} = \begin{pmatrix} 0 & 1 \\ \left(\frac{d}{dx}\right)^2 - q^2 & 0 \end{pmatrix} \mathbf{w} - \begin{pmatrix} 0 \\ 2\delta'(x) \end{pmatrix} u, \quad x \in \mathbb{R}, t \in (0, T), \tag{14}$$

$$\mathbf{w}(\cdot, 0) = \mathbf{w}^0 \to \mathbf{w}(\cdot, T) = \mathbf{w}^T, \tag{15}$$

where δ is the Dirac distribution, $\delta = H'$, H is the Heaviside function: $H(\xi) = 1$ if $\xi > 0$, and $H(\xi) = 0$ otherwise.
Further we consider the approximate null-controllability problem for the system (14) where $\mathbf{w}^0 \in \mathbf{H}^0$ and $\mathbf{w}^T \in \mathbf{H}^0$ are odd functions.

Fourier transform of the control system

Denote $\mathbf{y}(\cdot, t) = \mathcal{F}_{x\to\sigma}\begin{pmatrix}\mathbf{w}(\cdot, t)\\ \mathbf{w}_t(\cdot, t)\end{pmatrix}$, $\mathbf{y}^0 = \mathcal{F}\mathbf{w}^0$, $\mathbf{y}^T = \mathcal{F}\mathbf{w}^T$. Evidently, $\frac{d^m}{dt^m}\mathbf{y} : [0, T] \to \widetilde{H}_m \times \widetilde{H}_{m-1}$, $m = 0, 1$, $\mathbf{y}^0 \in \widetilde{H}_0 \times \widetilde{H}_{-1}$ and $\mathbf{y}^T \in \widetilde{H}_0 \times \widetilde{H}_{-1}$. Here $\mathbf{H}_m = \widetilde{H}_m \times \widetilde{H}_{m-1}$ with the norm $|\!|\!|\cdot|\!|\!|_m$, $m \in \mathbb{Z}$.

Fourier transform of the control system

Denote $\mathbf{y}(\cdot,t)=\mathcal{F}_{x\to\sigma}\begin{pmatrix}\mathbf{w}(\cdot,t)\\ \mathbf{w}_t(\cdot,t)\end{pmatrix}$, $\mathbf{y}^0=\mathcal{F}\mathbf{w}^0$, $\mathbf{y}^T=\mathcal{F}\mathbf{w}^T$. Evidently, $\frac{d^m}{dt^m}\mathbf{y}:[0,T]\to\widetilde{H}_m\times\widetilde{H}_{m-1}$, $m=0,1$, $\mathbf{y}^0\in\widetilde{H}_0\times\widetilde{H}_{-1}$ and $\mathbf{y}^T\in\widetilde{H}_0\times\widetilde{H}_{-1}$. Here $\mathbf{H}_m=\widetilde{H}_m\times\widetilde{H}_{m-1}$ with the norm $\left\|\cdot\right\|_m$, $m\in\mathbb{Z}$.

Applying to (14), (15) Fourier transform w.r.t. ξ, we obtain

$$\mathbf{y}_t=\begin{pmatrix}0 & 1\\ -\sigma^2-q^2 & 0\end{pmatrix}\mathbf{y}-\sqrt{\frac{2}{\pi}}\begin{pmatrix}0\\ i\sigma u(t)\end{pmatrix},\ \sigma\in\mathbb{R}, t\in(0,T), \qquad (16)$$

Fourier transform of the control system

Denote $\mathbf{y}(\cdot,t) = \mathcal{F}_{x\to\sigma}\begin{pmatrix}\mathbf{w}(\cdot,t)\\ \mathbf{w}_t(\cdot,t)\end{pmatrix}$, $\mathbf{y}^0 = \mathcal{F}\mathbf{w}^0$, $\mathbf{y}^T = \mathcal{F}\mathbf{w}^T$. Evidently, $\frac{d^m}{dt^m}\mathbf{y} : [0,T] \to \widetilde{H}_m \times \widetilde{H}_{m-1}$, $m = 0,1$, $\mathbf{y}^0 \in \widetilde{H}_0 \times \widetilde{H}_{-1}$ and $\mathbf{y}^T \in \widetilde{H}_0 \times \widetilde{H}_{-1}$. Here $\mathbf{H}_m = \widetilde{H}_m \times \widetilde{H}_{m-1}$ with the norm $|||\cdot|||_m$, $m \in \mathbb{Z}$.

Applying to (14), (15) Fourier transform w.r.t. ξ, we obtain

$$\mathbf{y}_t = \begin{pmatrix}0 & 1\\ -\sigma^2 - q^2 & 0\end{pmatrix}\mathbf{y} - \sqrt{\frac{2}{\pi}}\begin{pmatrix}0\\ i\sigma u(t)\end{pmatrix}, \ \sigma \in \mathbb{R}, t \in (0,T), \tag{16}$$

$$\mathbf{y}(\sigma,0) = \mathbf{y}^0(\sigma) \longrightarrow \mathbf{y}(\sigma,T) = \mathbf{y}^T(\sigma), \qquad \sigma \in \mathbb{R}, \tag{17}$$

Solutions to (16), (27)

We have

$$\mathbf{y}^T(\sigma) = \Sigma(\sigma, t)\left(\mathbf{y}^0(\sigma) - \sqrt{\frac{2}{\pi}}\int_0^T \begin{pmatrix} -\frac{\sin\left(t\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}} \\ \cos\left(t\sqrt{\sigma^2+q^2}\right) \end{pmatrix} u(t)\, dt\right)$$

Solutions to (16), (27)

We have

$$\mathbf{y}^T(\sigma) = \Sigma(\sigma,t)\left(\mathbf{y}^0(\sigma) - \sqrt{\frac{2}{\pi}}\int_0^T \begin{pmatrix} -\frac{\sin\left(t\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}} \\ \cos\left(t\sqrt{\sigma^2+q^2}\right)\end{pmatrix} u(t)\,dt\right)$$

where $\Sigma(\sigma,t) = \begin{pmatrix} \cos\left(t\sqrt{\sigma^2+q^2}\right) & \frac{\sin\left(t\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}} \\ -\sqrt{\sigma^2+q^2}\sin\left(t\sqrt{\sigma^2+q^2}\right) & \cos\left(t\sqrt{\sigma^2+q^2}\right)\end{pmatrix}$.

Solutions to (16), (27)

We have

$$\mathbf{y}^T(\sigma) = \Sigma(\sigma,t)\left(\mathbf{y}^0(\sigma) - \sqrt{\frac{2}{\pi}}\int_0^T \begin{pmatrix} -\frac{\sin\left(t\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}} \\ \cos\left(t\sqrt{\sigma^2+q^2}\right)\end{pmatrix} u(t)\,dt\right)$$

where $\Sigma(\sigma,t) = \begin{pmatrix} \cos\left(t\sqrt{\sigma^2+q^2}\right) & \frac{\sin\left(t\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}} \\ -\sqrt{\sigma^2+q^2}\sin\left(t\sqrt{\sigma^2+q^2}\right) & \cos\left(t\sqrt{\sigma^2+q^2}\right)\end{pmatrix}$.

We have

$$\|\Sigma(\cdot,t)\times\|_0 \le \begin{bmatrix} 1/q & \text{if } q>0 \\ 2\sqrt{1+t^2} & \text{if } q=0 \end{bmatrix}, \qquad t\in\mathbb{R}.$$

Operators Ψ and $\widehat{\Psi}$

Denote $\Psi : \widetilde{H}^0 \longrightarrow \widetilde{H}^0$ with $D(\Psi) = \widetilde{H}^0_0$ such that

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi).$$

Operators Ψ and $\widehat{\Psi}$

Denote $\Psi : \widetilde{H}^0 \longrightarrow \widetilde{H}^0$ with $D(\Psi) = \widetilde{H}^0_0$ such that

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi).$$

Denote $\widehat{\Psi} : \widetilde{H}^0 \longrightarrow \widetilde{H}^{-1}$ with $D(\widehat{\Psi}) = \widetilde{H}^0$ such that

$$\left(\widehat{\Psi}g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma\to x}\left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}).$$

Operators Ψ and $\widehat{\Psi}$

Denote $\Psi : \widetilde{H}^0 \longrightarrow \widetilde{H}^0$ with $D(\Psi) = \widetilde{H}^0_0$ such that

$$\left(\Psi g\right)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi).$$

Denote $\widehat{\Psi} : \widetilde{H}^0 \longrightarrow \widetilde{H}^{-1}$ with $D(\widehat{\Psi}) = \widetilde{H}^0$ such that

$$\left(\widehat{\Psi} g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma\to x}\left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}).$$

Evidently, if $q = 0$, then $\Psi = \mathrm{Id}$, $\widehat{\Psi} = \frac{d}{dx}(\operatorname{sgn}(\cdot))$.

Therefore

$$\mathbf{w}^T(x) = \mathbf{w}(x, T) = E(x, T) * \left[\mathbf{w}^0(x) - \begin{pmatrix} \Psi\mathcal{U} \\ \widehat{\Psi}\mathcal{U} \end{pmatrix}(x)\right] \qquad (18)$$

where $\mathcal{U}(t) = u(t)(H(t) - H(t - T)) - u(-t)(H(t + T) - H(-t))$, $t \in (0, +\infty)$, $*$ is the convolution with respect to x.

Therefore

$$\mathbf{w}^T(x) = \mathbf{w}(x,T) = E(x,T) * \left[\mathbf{w}^0(x) - \begin{pmatrix} \Psi\mathcal{U} \\ \widehat{\Psi}\mathcal{U} \end{pmatrix}(x)\right] \qquad (18)$$

where $\mathcal{U}(t) = u(t)(H(t) - H(t-T)) - u(-t)(H(t+T) - H(-t))$, $t \in (0, +\infty)$, $*$ is the convolution with respect to x.
Here

$$E(x,t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}_\sigma^{-1} \begin{pmatrix} \partial/\partial t & 1 \\ (\partial/\partial t)^2 & \partial/\partial t \end{pmatrix} \frac{\sin(t\sqrt{\sigma^2+q^2})}{\sqrt{\sigma^2+q^2}}$$
$$= \frac{1}{2} \begin{pmatrix} \partial/\partial t & 1 \\ (\partial/\partial t)^2 & \partial/\partial t \end{pmatrix} \left[\operatorname{sgn} t \; H\left(t^2 - x^2\right) J_0\left(q\sqrt{t^2-x^2}\right)\right]$$

Therefore

$$\mathbf{w}^T(x) = \mathbf{w}(x,T) = E(x,T) * \left[\mathbf{w}^0(x) - \begin{pmatrix} \Psi\mathcal{U} \\ \widehat{\Psi}\mathcal{U} \end{pmatrix}(x)\right] \qquad (18)$$

where $\mathcal{U}(t) = u(t)(H(t) - H(t-T)) - u(-t)(H(t+T) - H(-t))$, $t \in (0, +\infty)$, $*$ is the convolution with respect to x.
Here

$$E(x,t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}_\sigma^{-1} \begin{pmatrix} \partial/\partial t & 1 \\ (\partial/\partial t)^2 & \partial/\partial t \end{pmatrix} \frac{\sin(t\sqrt{\sigma^2+q^2})}{\sqrt{\sigma^2+q^2}}$$
$$= \frac{1}{2} \begin{pmatrix} \partial/\partial t & 1 \\ (\partial/\partial t)^2 & \partial/\partial t \end{pmatrix} \left[\operatorname{sgn} t \; H\left(t^2 - x^2\right) J_0\left(q\sqrt{t^2 - x^2}\right)\right]$$

where $J_k = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma(p+k+1)} \left(\frac{x}{2}\right)^{2p+k}$ is the Bessel function (here Γ is the Euler gamma function).

Since the Fourier transform operator $\mathcal{F}$ is an isomorphic isomorphism of H^m and H_m,
we have

$$\|E(\cdot,t)*\|^0 \le \begin{bmatrix} 1/q & \text{if } q>0 \\ 2\sqrt{1+t^2} & \text{if } q=0 \end{bmatrix}, \qquad t\in\mathbb{R}.$$

Uniqueness and well-posedness

Remark *It is well known that the solution to problem* (14), (15) *is unique.*

Uniqueness and well-posedness

Remark *It is well known that the solution to problem* (14), (15) *is unique.*

Remark *One can see that*

$$\left\| \begin{pmatrix} \mathbf{w}(\cdot, t) \\ \mathbf{w}_t(\cdot, t) \end{pmatrix} \right\|^0 \leq Q(T) \left(\left\| \mathbf{w}^0 \right\|^0 + \|u\|_{L^\infty(0,T)} \right), \qquad t \in [0, T],$$

where $Q(T) > 0$. *Therefore, problem* (14), (15) *is well posed.*

Null-controllability problems at a free time

According to definition,
a state $\mathbf{w}^0 \in \mathbf{H}^0$ is approximately null-controllable at a free time iff

$$\forall n \in \mathbb{N}\ \exists T_n > 0\ \exists u_n \in L^\infty(0, T_n) \quad \left\|\mathbf{w}^n(\cdot, T_n)\right\|^0 < 1/n, \tag{19}$$

where $\mathbf{w}^n$ is the solution of (14), (15) with $T = T_n$ and $u = u_n$.

Null-controllability problems at a free time

According to definition,
a state $\mathbf{w}^0 \in \mathbf{H}^0$ is approximately null-controllable at a free time iff

$$\forall n \in \mathbb{N}\ \exists T_n > 0\ \exists u_n \in L^\infty(0, T_n) \quad \left\|\mathbf{w}^n(\cdot, T_n)\right\|^0 < 1/n, \tag{19}$$

where $\mathbf{w}^n$ is the solution of (14), (15) with $T = T_n$ and $u = u_n$.
Put $\mathcal{U}_n(t) = u_n(t)(H(t) - H(t - T_n)) - u_n(-t)(H(t - T_n) - H(t))$, $n \in \mathbb{N}$.

Null-controllability problems at a free time

According to definition,
a state $\mathbf{w}^0 \in \mathbf{H}^0$ is approximately null-controllable at a free time iff

$$\forall n \in \mathbb{N}\ \exists T_n > 0\ \exists u_n \in L^\infty(0, T_n) \quad \left\| \mathbf{w}^n(\cdot, T_n) \right\|^0 < 1/n, \tag{19}$$

where $\mathbf{w}^n$ is the solution of (14), (15) with $T = T_n$ and $u = u_n$.
Put $\mathcal{U}_n(t) = u_n(t)(H(t) - H(t - T_n)) - u_n(-t)(H(t - T_n) - H(t))$, $n \in \mathbb{N}$.
Condition (19) is equivalent to

$$\forall n \in \mathbb{N}\ \exists T_n > 0\ \exists \mathcal{U}_n \in \widetilde{H}^0 \cap L^\infty(\mathbb{R}) \begin{cases} \operatorname{supp} \mathcal{U}_n \subset [-T_n, T_n] \\ \mathbf{w}_0^n = \Psi \mathcal{U}_n \to \mathbf{w}_0^0 \text{ as } n \to \infty \\ \mathbf{w}_1^n = \widehat{\Psi} \mathcal{U}_n \to \mathbf{w}_1^0 \text{ as } n \to \infty \end{cases},$$

Null-controllability problems at a free time

According to definition,
a state $\mathbf{w}^0 \in \mathbf{H}^0$ is approximately null-controllable at a free time iff

$$\forall n \in \mathbb{N}\ \exists T_n > 0\ \exists u_n \in L^\infty(0, T_n) \quad \left\| \mathbf{w}^n(\cdot, T_n) \right\|^0 < 1/n, \tag{19}$$

where $\mathbf{w}^n$ is the solution of (14), (15) with $T = T_n$ and $u = u_n$.
Put $\mathcal{U}_n(t) = u_n(t)(H(t) - H(t - T_n)) - u_n(-t)(H(t - T_n) - H(t))$, $n \in \mathbb{N}$.
Condition (19) is equivalent to

$$\forall n \in \mathbb{N}\ \exists T_n > 0\ \exists \mathcal{U}_n \in \widetilde{H}^0 \cap L^\infty(\mathbb{R}) \begin{cases} \operatorname{supp} \mathcal{U}_n \subset [-T_n, T_n] \\ \mathbf{w}_0^n = \Psi \mathcal{U}_n \to \mathbf{w}_0^0 \text{ as } n \to \infty \\ \mathbf{w}_1^n = \widehat{\Psi} \mathcal{U}_n \to \mathbf{w}_1^0 \text{ as } n \to \infty \end{cases},$$

$$E(x, -T_n) * \mathbf{w}^n(x, T_n) = \mathbf{w}^0(x) - \begin{pmatrix} \Psi \mathcal{U}_n \\ \widehat{\Psi} \mathcal{U}_n \end{pmatrix}(x).$$

Difference between the cases $q = 0$ and $q > 0$

$q = 0$:

$$\begin{cases} \mathbf{w}_0^n = \Psi \mathcal{U}_n = \mathcal{U}_n \to \mathbf{w}_0^0 \text{ as } n \to \infty \\ \mathbf{w}_1^n = \widehat{\Psi} \mathcal{U}_n = (\operatorname{sgn}(\cdot)\, \mathcal{U}_n)' \to \mathbf{w}_1^0 \text{ as } n \to \infty \\ \qquad\qquad\qquad\qquad\qquad \searrow \quad \| \\ \qquad\qquad\qquad\qquad\qquad (\operatorname{sgn}(\cdot)\, \mathbf{w}_0^0)' \end{cases}$$

Difference between the cases $q = 0$ and $q > 0$

$q = 0$:

$$\begin{cases} \mathbf{w}_0^n = \Psi\mathcal{U}_n = \mathcal{U}_n \to \mathbf{w}_0^0 \text{ as } n \to \infty \\ \mathbf{w}_1^n = \widehat{\Psi}\mathcal{U}_n = (\operatorname{sgn}(\cdot)\,\mathcal{U}_n)' \to \mathbf{w}_1^0 \text{ as } n \to \infty \\ \qquad\qquad\qquad\qquad\qquad \searrow \quad \| \\ \qquad\qquad\qquad\qquad\qquad (\operatorname{sgn}(\cdot)\,\mathbf{w}_0^0)' \end{cases}$$

$q > 0$:

?

Properties of the operators Ψ and $\widehat{\Psi}$

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi) = \widetilde{H}^0,$$

$$\left(\widehat{\Psi}g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma\to x}\left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}) = \widetilde{H}^0.$$

Properties of the operators Ψ and $\widehat{\Psi}$

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi) = \widetilde{H}^0,$$

$$\left(\widehat{\Psi}g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma\to x}\left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}) = \widetilde{H}^0.$$

For $q = 0$, $\Psi = \mathrm{Id}$, $\widehat{\Psi} = \frac{d}{dx}(\operatorname{sgn}(\cdot))$, and their properties are evident.

Properties of the operators Ψ and $\widehat{\Psi}$

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma \to x} \left(\frac{\sigma \left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}} \right)(x), \quad g \in D(\Psi) = \widetilde{H}^0,$$

$$\left(\widehat{\Psi} g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma \to x} \left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}) = \widetilde{H}^0.$$

For $q = 0$, $\Psi = \mathrm{Id}$, $\widehat{\Psi} = \frac{d}{dx}(\operatorname{sgn}(\cdot))$, and their properties are evident.

Theorem

Let $q > 0$.

- Ψ *and* $\widehat{\Psi}$ *are bounded;*

Properties of the operators Ψ and $\widehat{\Psi}$

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi) = \widetilde{H}^0,$$

$$\left(\widehat{\Psi}g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma\to x}\left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}) = \widetilde{H}^0.$$

For $q = 0$, $\Psi = \mathrm{Id}$, $\widehat{\Psi} = \frac{d}{dx}(\operatorname{sgn}(\cdot))$, and their properties are evident.

Theorem

Let $q > 0$.

- Ψ *and* $\widehat{\Psi}$ *are bounded;*
- $\overline{R(\Psi)} = \widetilde{H}^0$ *and* $\overline{R(\widehat{\Psi})} = \widetilde{H}^{-1}$*;*

Properties of the operators Ψ and $\widehat{\Psi}$

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi) = \widetilde{H}^0,$$

$$\left(\widehat{\Psi}g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma\to x}\left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}) = \widetilde{H}^0.$$

For $q = 0$, $\Psi = \mathrm{Id}$, $\widehat{\Psi} = \frac{d}{dx}(\operatorname{sgn}(\cdot))$, and their properties are evident.

Theorem

Let $q > 0$.

- Ψ *and* $\widehat{\Psi}$ *are bounded;*
- $\overline{R(\Psi)} = \widetilde{H}^0$ *and* $\overline{R(\widehat{\Psi})} = \widetilde{H}^{-1}$*;*
- $N(\Psi) = \left\{g \in \widetilde{H}^0_0 \mid \mathcal{F}g \subset [-q, q]\right\}$*;*

Properties of the operators Ψ and $\widehat{\Psi}$

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi) = \widetilde{H}^0,$$

$$\left(\widehat{\Psi} g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma\to x}\left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}) = \widetilde{H}^0.$$

For $q = 0$, $\Psi = \mathrm{Id}$, $\widehat{\Psi} = \frac{d}{dx}(\operatorname{sgn}(\cdot))$, and their properties are evident.

Theorem

Let $q > 0$.

- Ψ *and* $\widehat{\Psi}$ *are bounded;*
- $\overline{R(\Psi)} = \widetilde{H}^0$ *and* $\overline{R(\widehat{\Psi})} = \widetilde{H}^{-1}$*;*
- $N(\Psi) = \left\{g \in \widetilde{H}^0_0 \mid \mathcal{F}g \subset [-q, q]\right\}$*;*
- $N(\widehat{\Psi}) = \left\{g \in \widetilde{H}^0_0 \mid \mathcal{F}(\operatorname{sgn} t\, g) \subset [-q, q]\right\}$*;*

Properties of the operators Ψ and $\widehat{\Psi}$

$$(\Psi g)(x) = \mathcal{F}^{-1}_{\sigma\to x}\left(\frac{\sigma\left(\mathcal{F}g\right)\left(\sqrt{\sigma^2+q^2}\right)}{\sqrt{\sigma^2+q^2}}\right)(x), \quad g \in D(\Psi) = \widetilde{H}^0,$$

$$\left(\widehat{\Psi}g\right)(x) = \frac{d}{dx}\mathcal{F}^{-1}_{\sigma\to x}\left(\left(\mathcal{F}\left(\operatorname{sgn}\xi\, g\right)\right)\left(\sqrt{\sigma^2+q^2}\right)\right)(x),\ g \in D(\widehat{\Psi}) = \widetilde{H}^0.$$

For $q = 0$, $\Psi = \mathrm{Id}$, $\widehat{\Psi} = \frac{d}{dx}(\operatorname{sgn}(\cdot))$, and their properties are evident.

Theorem

Let $q > 0$.

- Ψ *and* $\widehat{\Psi}$ *are bounded;*
- $\overline{R(\Psi)} = \widetilde{H}^0$ *and* $\overline{R(\widehat{\Psi})} = \widetilde{H}^{-1}$*;*
- $N(\Psi) = \left\{g \in \widetilde{H}^0_0 \mid \mathcal{F}g \subset [-q, q]\right\}$*;*
- $N(\widehat{\Psi}) = \left\{g \in \widetilde{H}^0_0 \mid \mathcal{F}(\operatorname{sgn} t\, g) \subset [-q, q]\right\}$*;*
- Ψ *and* $\widehat{\Psi}$ *are not invertible.*

Properties of $\overline{\widehat{\Psi}(N(\Psi))}$ and $\overline{\Psi(N(\widehat{\Psi}))}$

Theorem (L.V.Fardigola, ESAIM: COCV 18 (2012), 748–773)

Let $q > 0$, $n = \overline{0, \infty}$. Then $\operatorname{sgn} x\, |x|^n e^{-q|x|} \in \overline{\widehat{\Psi}(N(\Psi))}$ *(the closure is considered in $\widetilde{H}^{-1}$).*

Properties of $\overline{\widehat{\Psi}(N(\Psi))}$ and $\overline{\Psi(N(\widehat{\Psi}))}$

Theorem (L.V.Fardigola, ESAIM: COCV 18 (2012), 748–773)

Let $q > 0$, $n = \overline{0, \infty}$. *Then* $\operatorname{sgn} x\, |x|^n e^{-q|x|} \in \overline{\widehat{\Psi}(N(\Psi))}$ *(the closure is considered in* $\widetilde{H}^{-1}$*).*

Theorem (L.V.Fardigola, ESAIM: COCV 18 (2012), 748–773)

Let $q > 0$, $n = \overline{0, \infty}$. *Then* $\operatorname{sgn} x\, |x|^n e^{-q|x|} \in \overline{\Psi(N(\widehat{\Psi}))}$ *(the closure is considered in* $\widetilde{H}^{0}$*).*

Properties of $\overline{\widehat{\Psi}(N(\Psi))}$ and $\overline{\Psi(N(\widehat{\Psi}))}$

Since the system of elements $\{\operatorname{sgn} x\, |x|^n e^{-q|x|}\}_{n=0}^{\infty}$ is closed in $\widetilde{H}^0$ and $\widetilde{H}^{-1}$, we have two theorems:

Theorem (L.V.Fardigola, ESAIM: COCV 18 (2012), 748–773)

Let $q > 0$. Then $\widetilde{H}^{-1}$ is the closure of $\widehat{\Psi}(N(\Psi))$ with respect to to the norm $\|\cdot\|^{-1}$.

Properties of $\overline{\widehat{\Psi}(N(\Psi))}$ and $\overline{\Psi(N(\widehat{\Psi}))}$

Since the system of elements $\{\operatorname{sgn} x\,|x|^n e^{-q|x|}\}_{n=0}^{\infty}$ is closed in $\widetilde{H}^0$ and $\widetilde{H}^{-1}$, we have two theorems:

Theorem (L.V.Fardigola, ESAIM: COCV 18 (2012), 748–773)

Let $q > 0$. Then $\widetilde{H}^{-1}$ is the closure of $\widehat{\Psi}(N(\Psi))$ with respect to to the norm $\|\cdot\|^{-1}$.

Theorem (L.V.Fardigola, ESAIM: COCV 18 (2012), 748–773)

Let $q > 0$. Then $\widetilde{H}^0$ is the closure of $\overline{\Psi(N(\widehat{\Psi}))}$ with respect to to the norm $\|\cdot\|^0$.

Approximate null-controllability problems at a free time

Let $q > 0$, $\mathbf{w}^0 = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix}$.

Approximate null-controllability problems at a free time

Let $q > 0$, $\mathbf{w}^0 = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix}$.

$$\widetilde{H}^0 = \overline{\Psi(N(\widehat{\Psi}))} \Leftrightarrow \exists \{g_0^n\}_{n=1}^{\infty} \subset N(\widehat{\Psi}) \begin{cases} \mathbf{w}_0^n = \Psi g_0^n \to \mathbf{w}_0^0 \\ 0 = \widehat{\Psi} g_0^n \to 0 \end{cases} \text{as } n \to \infty,$$

$$\widetilde{H}^{-1} = \overline{\widehat{\Psi}(N(\Psi))} \Leftrightarrow \exists \{g_1^n\}_{n=1}^{\infty} \subset N(\Psi) \begin{cases} 0 = \Psi g_1^n \to 0 \\ \mathbf{w}_1^n = \widehat{\Psi} g_1^n \to \mathbf{w}_1^0 \end{cases} \text{as } n \to \infty.$$

Approximate null-controllability problems at a free time

Let $q > 0$, $\mathbf{w}^0 = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix}$.

$$\widetilde{H}^0 = \overline{\Psi(N(\widehat{\Psi}))} \Leftrightarrow \exists \{g_0^n\}_{n=1}^\infty \subset N(\widehat{\Psi}) \begin{cases} \mathbf{w}_0^n = \Psi g_0^n \to \mathbf{w}_0^0 \\ 0 = \widehat{\Psi} g_0^n \to 0 \end{cases} \text{as } n \to \infty,$$

$$\widetilde{H}^{-1} = \overline{\widehat{\Psi}(N(\Psi))} \Leftrightarrow \exists \{g_1^n\}_{n=1}^\infty \subset N(\Psi) \begin{cases} 0 = \Psi g_1^n \to 0 \\ \mathbf{w}_1^n = \widehat{\Psi} g_1^n \to \mathbf{w}_1^0 \end{cases} \text{as } n \to \infty.$$

For $g^n = g_0^n + g_1^n$, $n \in \mathbb{N}$, we have $g^n \in \widetilde{H}^0$, $n \in \mathbb{N}$, and

$$\begin{cases} \Psi g^n = \Psi g_0^n \to \mathbf{w}_0^0 \\ \widehat{\Psi} g^n = \Psi g_1^n \to \mathbf{w}_1^0 \end{cases} \text{as } n \to \infty.$$

Approximate null-controllability problems at a free time

We can find a sequence $\{\mathcal{U}^n\}_{n=0}^{\infty} \subset \widetilde{H}^0 \cap L^\infty(\mathbb{R})$ such that $\operatorname{supp} \mathcal{U}^n \subset [-T_n, T_n]$, $n \in \mathbb{N}$, and

$$\|g^n - \mathcal{U}^n\|^0 \to 0 \quad \text{as } n \to \infty.$$

Approximate null-controllability problems at a free time

We can find a sequence $\{\mathcal{U}^n\}_{n=0}^{\infty} \subset \widetilde{H}^0 \cap L^\infty(\mathbb{R})$ such that $\operatorname{supp}\mathcal{U}^n \subset [-T_n, T_n]$, $n \in \mathbb{N}$, and

$$\|g^n - \mathcal{U}^n\|^0 \to 0 \quad \text{as } n \to \infty.$$

Let $\mathbf{w}^n$ be the solution to control system (14), (15) with $T = T_n$ and $u(t) = \mathcal{U}^n(t)$, $t \in [0, T_n]$, $n \in \mathbb{N}$.

Approximate null-controllability problems at a free time

We can find a sequence $\{\mathcal{U}^n\}_{n=0}^{\infty} \subset \widetilde{H}^0 \cap L^\infty(\mathbb{R})$ such that $\operatorname{supp}\mathcal{U}^n \subset [-T_n, T_n]$, $n \in \mathbb{N}$, and

$$\|g^n - \mathcal{U}^n\|^0 \to 0 \quad \text{as } n \to \infty.$$

Let $\mathbf{w}^n$ be the solution to control system (14), (15) with $T = T_n$ and $u(t) = \mathcal{U}^n(t)$, $t \in [0, T_n]$, $n \in \mathbb{N}$.
Since the operators Ψ and $\widehat{\Psi}$ are bounded, we have

$$\left\|\mathbf{w}^T\right\|^0 \le \frac{1}{q}\left\|\mathbf{w}^0 - \begin{pmatrix}\Psi\mathcal{U}^n\\ \widehat{\Psi}\mathcal{U}^n\end{pmatrix}\right\|^0$$
$$\le \frac{1}{q}\left(\left\|\mathbf{w}^0 - \begin{pmatrix}\Psi g^n\\ \widehat{\Psi} g^n\end{pmatrix}\right\|^0 + \left\|\begin{pmatrix}\Psi(g^n - \mathcal{U}^n)\\ \widehat{\Psi}(g^n - \mathcal{U}^n)\end{pmatrix}\right\|^0\right) \to 0 \quad \text{as } n \to \infty.$$

Approximate null-controllability problems at a free time

We can find a sequence $\{\mathcal{U}^n\}_{n=0}^{\infty} \subset \widetilde{H}^0 \cap L^\infty(\mathbb{R})$ such that $\operatorname{supp}\mathcal{U}^n \subset [-T_n, T_n]$, $n \in \mathbb{N}$, and

$$\|g^n - \mathcal{U}^n\|^0 \to 0 \quad \text{as } n \to \infty.$$

Let $\mathbf{w}^n$ be the solution to control system (14), (15) with $T = T_n$ and $u(t) = \mathcal{U}^n(t)$, $t \in [0, T_n]$, $n \in \mathbb{N}$.
Since the operators Ψ and $\widehat{\Psi}$ are bounded, we have

$$\left\|\left\|\mathbf{w}^T\right\|\right\|^0 \le \frac{1}{q}\left\|\left\|\mathbf{w}^0 - \begin{pmatrix}\Psi\mathcal{U}^n\\ \widehat{\Psi}\mathcal{U}^n\end{pmatrix}\right\|\right\|^0 \le \frac{1}{q}\left(\left\|\left\|\mathbf{w}^0 - \begin{pmatrix}\Psi g^n\\ \widehat{\Psi} g^n\end{pmatrix}\right\|\right\|^0 + \left\|\left\|\begin{pmatrix}\Psi(g^n - \mathcal{U}^n)\\ \widehat{\Psi}(g^n - \mathcal{U}^n)\end{pmatrix}\right\|\right\|^0\right) \to 0 \quad \text{as } n \to \infty.$$

$$\mathbf{w}(x, T_n) = E(x, T_n) * \left[\mathbf{w}^0(x) - \begin{pmatrix}\Psi\mathcal{U}^n\\ \widehat{\Psi}\mathcal{U}^n\end{pmatrix}(x)\right] \text{ and } \|E(\cdot, T_n)*\|^0 \le \frac{1}{q}.$$

Necessary and sufficient conditions for approximate null-controllability at a free time

Thus we obtain the following theorem

Theorem (L.V.Fardigola, ESAIM: COCV 18 (2012), 748–773)

Let $q > 0$. Each state $\mathbf{w}^0 \in \mathbf{H}$ is approximately null-controllable at a ***free*** *time.*

Necessary and sufficient conditions for approximate null-controllability at a free time

Thus we obtain the following theorem

Theorem (L.V.Fardigola, ESAIM: COCV 18 (2012), 748–773)

*Let $q > 0$. Each state $\mathbf{w}^0 \in \mathbf{H}$ is approximately null-controllable at a **free** time.*

By analysing the d'Alembert formula for the solution of the wave equation, we obtain the following theorem

Theorem (L.V.Fardigola and G.M.Sklyar, JMAA 276(2002), No. 2, 109–134)

*Let $q = 0$. A state $\mathbf{w}^0 \in \mathbf{H}$ is approximately null-controllable at a **free** time iff*

$$\mathbf{w}_1^0 - \left(\operatorname{sgn} x\, \mathbf{w}_0^0\right)' = 0. \tag{20}$$

Example

Let $q > 0$, $\mathbf{w}_0^0(x) = e^{-q|x|} \operatorname{sgn} x$, $\mathbf{w}_1^0(x) = 0$, $x \in \mathbb{R}$.

Example

Let $q > 0$, $\mathbf{w}_0^0(x) = e^{-q|x|} \operatorname{sgn} x$, $\mathbf{w}_1^0(x) = 0$, $x \in \mathbb{R}$.
Consider the following control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u(t)\delta'(x), \qquad x \in \mathbb{R},\ t \in (0, T),$$
$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0^0, \qquad \mathbf{w}_t(\cdot, 0) = \mathbf{w}_1^0.$$

Example

Let $q > 0$, $\mathbf{w}_0^0(x) = e^{-q|x|}\operatorname{sgn} x$, $\mathbf{w}_1^0(x) = 0$, $x \in \mathbb{R}$.
Consider the following control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u(t)\delta'(x), \qquad x \in \mathbb{R},\ t \in (0, T),$$
$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0^0, \qquad \mathbf{w}_t(\cdot, 0) = \mathbf{w}_1^0.$$

For $n \geq \frac{\sqrt{2}}{q}$, set $T_n = n^6$, $u_n(t) = n\frac{\sin(t/n)}{t}$, $t \in [0, T_n]$.

Example

Let $q > 0$, $\mathbf{w}_0^0(x) = e^{-q|x|} \operatorname{sgn} x$, $\mathbf{w}_1^0(x) = 0$, $x \in \mathbb{R}$.
Consider the following control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u(t)\delta'(x), \qquad x \in \mathbb{R},\ t \in (0, T),$$
$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0^0, \qquad \mathbf{w}_t(\cdot, 0) = \mathbf{w}_1^0.$$

For $n \geq \frac{\sqrt{2}}{q}$, set $T_n = n^6$, $u_n(t) = n\frac{\sin(t/n)}{t}$, $t \in [0, T_n]$.
Let $\mathbf{w}^n$ be the solution to the control problem with $T = T_n$ and $u = u_n$.

Example

Let $q > 0$, $\mathbf{w}_0^0(x) = e^{-q|x|} \operatorname{sgn} x$, $\mathbf{w}_1^0(x) = 0$, $x \in \mathbb{R}$.
Consider the following control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u(t)\delta'(x), \qquad x \in \mathbb{R},\ t \in (0, T),$$
$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0^0, \qquad \mathbf{w}_t(\cdot, 0) = \mathbf{w}_1^0.$$

For $n \geq \frac{\sqrt{2}}{q}$, set $T_n = n^6$, $u_n(t) = n\frac{\sin(t/n)}{t}$, $t \in [0, T_n]$.
Let $\mathbf{w}^n$ be the solution to the control problem with $T = T_n$ and $u = u_n$.
Then

$$\left\| \begin{pmatrix} \mathbf{w}^n(\cdot, T_n) \\ \mathbf{w}_t(\cdot, T_n) \end{pmatrix} \right\|^0 \leq \frac{1 + 2q^{5/2}}{q^{5/2} n^2} \to 0 \quad \text{as } n \to \infty.$$

Example

Let $q > 0$, $\mathbf{w}_0^0(x) = e^{-q|x|}\operatorname{sgn} x$, $\mathbf{w}_1^0(x) = 0$, $x \in \mathbb{R}$.
Consider the following control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u(t)\delta'(x), \qquad x \in \mathbb{R},\ t \in (0, T),$$
$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0^0, \qquad \mathbf{w}_t(\cdot, 0) = \mathbf{w}_1^0.$$

For $n \geq \frac{\sqrt{2}}{q}$, set $T_n = n^6$, $u_n(t) = n\frac{\sin(t/n)}{t}$, $t \in [0, T_n]$.
Let $\mathbf{w}^n$ be the solution to the control problem with $T = T_n$ and $u = u_n$.
Then

$$\left\|\!\left\| \begin{pmatrix} \mathbf{w}^n(\cdot, T_n) \\ \mathbf{w}_t(\cdot, T_n) \end{pmatrix} \right\|\!\right\|^0 \leq \frac{1 + 2q^{5/2}}{q^{5/2} n^2} \to 0 \quad \text{as } n \to \infty.$$

Thus the state $\mathbf{w}^0 = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix}$ is approximately null-controllable at a free time.

Example

Let $q > 0$, $\mathbf{w}_0^0(x) = e^{-q|x|} \operatorname{sgn} x$, $\mathbf{w}_1^0(x) = 0$, $x \in \mathbb{R}$.
Consider the following control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2 \mathbf{w} - 2u(t)\delta'(x), \qquad x \in \mathbb{R},\ t \in (0, T),$$
$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0^0, \qquad \mathbf{w}_t(\cdot, 0) = \mathbf{w}_1^0.$$

For $n \geq \frac{\sqrt{2}}{q}$, set $T_n = n^6$, $u_n(t) = n\frac{\sin(t/n)}{t}$, $t \in [0, T_n]$.
Let $\mathbf{w}^n$ be the solution to the control problem with $T = T_n$ and $u = u_n$.
Then

$$\left\| \begin{pmatrix} \mathbf{w}^n(\cdot, T_n) \\ \mathbf{w}_t(\cdot, T_n) \end{pmatrix} \right\|^0 \leq \frac{1 + 2q^{5/2}}{q^{5/2} n^2} \to 0 \quad \text{as } n \to \infty.$$

Thus the state $\mathbf{w}^0 = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix}$ is approximately null-controllable at a free time.
Moreover, the pairs (T_n, u_n), $n \geq \frac{\sqrt{2}}{q}$, solve the approximate null-controllability problem at a free time.

Null-controllability problems for the wave equation with variable coefficients

Now we consider the following controllability problem

$$z_{tt} = \frac{1}{\rho(\xi)} \left(k(\xi) z_\xi\right)_\xi + \gamma(\xi) z, \ \xi > 0, \ t \in (0, T), \tag{21}$$

Null-controllability problems for the wave equation with variable coefficients

Now we consider the following controllability problem

$$z_{tt} = \frac{1}{\rho(\xi)}\left(k(\xi)z_\xi\right)_\xi + \gamma(\xi)z,\ \ \xi > 0,\ \ t \in (0, T), \tag{21}$$

$$z(0, t) = v(t), \quad t \in (0, T), \tag{22}$$

Null-controllability problems for the wave equation with variable coefficients

Now we consider the following controllability problem

$$z_{tt} = \frac{1}{\rho(\xi)}\left(k(\xi)z_{\xi}\right)_{\xi} + \gamma(\xi)z,\ \ \xi > 0,\ \ t \in (0, T), \tag{21}$$

$$z(0,t) = v(t), \quad t \in (0, T), \tag{22}$$

$$\left.\begin{array}{r} z(\xi,0) = z_0^0(\xi) \\ z_t(\xi,0) = z_1^0(\xi) \end{array}\right\} \longrightarrow \left\{\begin{array}{l} z(\xi, T) = z_0^T(\xi) \\ z_t(\xi, T) = z_1^T(\xi) \end{array}\right. \tag{23}$$

Null-controllability problems for the wave equation with variable coefficients

Now we consider the following controllability problem

$$z_{tt} = \frac{1}{\rho(\xi)} \left(k(\xi) z_\xi\right)_\xi + \gamma(\xi) z, \ \xi > 0, \ t \in (0, T), \tag{21}$$

$$z(0, t) = v(t), \quad t \in (0, T), \tag{22}$$

$$\left.\begin{array}{r} z(\xi, 0) = z_0^0(\xi) \\ z_t(\xi, 0) = z_1^0(\xi) \end{array}\right\} \longrightarrow \left\{\begin{array}{l} z(\xi, T) = z_0^T(\xi) \\ z_t(\xi, T) = z_1^T(\xi) \end{array}\right. \tag{23}$$

where $T > 0$ is a constant; ρ, k, γ, w_0^0, and w_1^0 are given functions; $v \in L^\infty(0, T)$ is a control; $\rho, k, \gamma \in C^1[0, +\infty)$, ρ, k are positive on $[0, +\infty)$.

Assume $\eta = (k\rho)^{1/4}$, $\eta \in C^2(\mathbb{R})$, $\theta = (k/\rho)^{1/4}$,

Assume $\eta = (k\rho)^{1/4}$, $\eta \in C^2(\mathbb{R})$, $\theta = (k/\rho)^{1/4}$,

$$\sigma(\xi) = \int_0^{\xi} \frac{d\mu}{\theta^2(\mu)}, \quad \xi \in \mathbb{R}, \qquad \text{and} \qquad \sigma(\xi) \to +\infty \text{ as } \xi \to +\infty,$$

Assume $\eta = (k\rho)^{1/4}$, $\eta \in C^2(\mathbb{R})$, $\theta = (k/\rho)^{1/4}$,

$$\sigma(\xi) = \int_0^\xi \frac{d\mu}{\theta^2(\mu)}, \quad \xi \in \mathbb{R}, \qquad \text{and} \qquad \sigma(\xi) \to +\infty \text{ as } \xi \to +\infty,$$

$$\mathcal{D}_{\eta\theta} = \theta^2 \left(\frac{d}{dx} + \frac{\eta'}{\eta} \right).$$

Assume $\eta = (k\rho)^{1/4}$, $\eta \in C^2(\mathbb{R})$, $\theta = (k/\rho)^{1/4}$,

$$\sigma(\xi) = \int_0^{\xi} \frac{d\mu}{\theta^2(\mu)}, \quad \xi \in \mathbb{R}, \qquad \text{and} \qquad \sigma(\xi) \to +\infty \text{ as } \xi \to +\infty,$$

$$\mathcal{D}_{\eta\theta} = \theta^2 \left(\frac{d}{dx} + \frac{\eta'}{\eta} \right).$$

Then

$$\frac{1}{\rho}\left(kf'\right)' = \mathcal{D}_{\eta\theta}^2 f - \left(\mathcal{D}_{\eta\theta}\left(\theta^2 \frac{\eta'}{\eta}\right)\right) f.$$

Assume $\eta = (k\rho)^{1/4}$, $\eta \in C^2(\mathbb{R})$, $\theta = (k/\rho)^{1/4}$,

$$\sigma(\xi) = \int_0^\xi \frac{d\mu}{\theta^2(\mu)}, \quad \xi \in \mathbb{R}, \qquad \text{and} \qquad \sigma(\xi) \to +\infty \text{ as } \xi \to +\infty,$$

$$\mathcal{D}_{\eta\theta} = \theta^2 \left(\frac{d}{dx} + \frac{\eta'}{\eta} \right).$$

Then

$$\frac{1}{\rho} \left(k f' \right)' = \mathcal{D}^2_{\eta\theta} f - \left(\mathcal{D}_{\eta\theta} \left(\theta^2 \frac{\eta'}{\eta} \right) \right) f.$$

Let $\widehat{\gamma}$ be the even extension of γ, $p = \mathcal{D}_{\eta\theta} \left(\theta^2 \frac{\eta'}{\eta} \right)$.

Assume $\eta = (k\rho)^{1/4}$, $\eta \in C^2(\mathbb{R})$, $\theta = (k/\rho)^{1/4}$,

$$\sigma(\xi) = \int_0^\xi \frac{d\mu}{\theta^2(\mu)}, \quad \xi \in \mathbb{R}, \qquad \text{and} \qquad \sigma(\xi) \to +\infty \text{ as } \xi \to +\infty,$$

$$\mathcal{D}_{\eta\theta} = \theta^2 \left(\frac{d}{dx} + \frac{\eta'}{\eta} \right).$$

Then

$$\frac{1}{\rho} \left(kf'\right)' = \mathcal{D}^2_{\eta\theta} f - \left(\mathcal{D}_{\eta\theta} \left(\theta^2 \frac{\eta'}{\eta} \right) \right) f.$$

Let $\widehat{\gamma}$ be the even extension of γ, $p = \mathcal{D}_{\eta\theta} \left(\theta^2 \frac{\eta'}{\eta} \right)$.
We assume also that

$$\exists q = \text{const} \geq 0 \quad \left(r = p \circ \sigma^{-1} - q^2 \in C^1[0, +\infty) \cap L^2(0, +\infty) \right.$$
$$\left. \text{and } \int_0^\infty \lambda |r(\lambda)| \, d\lambda < \infty \right).$$

Spaces H^m and $\mathbb{H}^m$

Classical Sobolev spaces

$$H^p = \{\varphi \in L^2(\mathbb{R}) \mid \forall k = \overline{0,p}\ \frac{d^k}{dx^k}\varphi \in L^2(\mathbb{R})\},$$

$$\|\varphi\|^p = \left(\sum_{k=0}^{p} \left\|\frac{d^k}{dx^k}\varphi\right\|^2_{L^2(\mathbb{R})}\right)^{1/2},$$

$$H^{-p} = (H^p)^*,$$

$$\|f\|^{-p} = \sup\left\{\frac{|\langle f,\varphi\rangle|}{\|\varphi\|^p} \mid \|\varphi\|^p \neq 0\right\},$$

$$\left\langle \frac{d}{dx}f,\varphi\right\rangle = -\left\langle f,\frac{d}{dx}\varphi\right\rangle,\ \ p \neq 2.$$

Modified Sobolev spaces

$$\mathbb{H}^p = \{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}) \mid \forall k = \overline{0,p}\ \mathcal{D}^k_{\eta\theta}\psi \in L^2_{\eta\theta}(\mathbb{R})\},$$

$$[\![\psi]\!]^p = \left(\sum_{k=0}^{p} \left(\left\|\mathcal{D}^k_{\eta\theta}\psi\right)\right\|^2_{L^2_{\eta\theta}(\mathbb{R})}\right)^{1/2},$$

$$\mathbb{H}^{-p} = (\mathbb{H}^p)^*,$$

$$[\![g]\!]^{-p} = \sup\left\{\frac{|\langle\langle g,\psi\rangle\rangle|}{[\![\psi]\!]^p} \mid [\![\psi]\!]^p \neq 0\right\},$$

$$\langle\langle \mathcal{D}_{\eta\theta}g,\psi\rangle\rangle = -\langle\langle g,\mathcal{D}_{\eta\theta}\psi\rangle\rangle,\ \ p \neq 2.$$

Reduced control problem

Put $\widetilde{\mathbb{H}}^m = \{\varphi \in \mathbb{H}^m : \varphi \text{ is odd}\}$, $-2 \le m \le 2$, $\mathbb{H} = \widetilde{\mathbb{H}}^0 \times \widetilde{\mathbb{H}}^{-1}$ with the norm $[\![\cdot]\!]$.

Reduced control problem

Put $\widetilde{\mathbb{H}}^m = \{\varphi \in \mathbb{H}^m : \varphi \text{ is odd}\}$, $-2 \le m \le 2$, $\mathbb{H\!I} = \widetilde{\mathbb{H}}^0 \times \widetilde{\mathbb{H}}^{-1}$ with the norm $[\![\cdot]\!]$.

Let $\mathbf{z}(\cdot, t)$, $\mathbf{z}^0$, $\mathbf{z}^T$ be the odd extension w.r.t. ξ for $z(\cdot, t)$, $\begin{pmatrix} z_0^0 \\ z_1^0 \end{pmatrix}$, $\begin{pmatrix} z_0^T \\ z_1^T \end{pmatrix}$, resp., $(t \in [0, T])$.

Reduced control problem

Put $\widetilde{\mathbb{H}}^m = \{\varphi \in \mathbb{H}^m : \varphi \text{ is odd}\}$, $-2 \le m \le 2$, $\mathbb{H\!H} = \widetilde{\mathbb{H}}^0 \times \widetilde{\mathbb{H}}^{-1}$ with the norm $[\![\cdot]\!]$.

Let $\mathbf{z}(\cdot, t)$, $\mathbf{z}^0$, $\mathbf{z}^T$ be the odd extension w.r.t. ξ for $z(\cdot, t)$, $\begin{pmatrix} z_0^0 \\ z_1^0 \end{pmatrix}$, $\begin{pmatrix} z_0^T \\ z_1^T \end{pmatrix}$, resp., ($t \in [0, T]$).

Controllability problem (21)–(23) can be reduced to the following one

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p\mathbf{z} - 2\eta^2(0) v \mathcal{D}_{\eta\theta}\delta, \quad \xi \in \mathbb{R}, \ t \in (0, T), \tag{24}$$

Reduced control problem

Put $\widetilde{\mathbb{H}}^m = \{\varphi \in \mathbb{H}^m : \varphi \text{ is odd}\}$, $-2 \le m \le 2$, $\mathbb{HI} = \widetilde{\mathbb{H}}^0 \times \widetilde{\mathbb{H}}^{-1}$ with the norm $[\![\cdot]\!]$.

Let $\mathbf{z}(\cdot, t)$, $\mathbf{z}^0$, $\mathbf{z}^T$ be the odd extension w.r.t. ξ for $z(\cdot, t)$, $\begin{pmatrix} z_0^0 \\ z_1^0 \end{pmatrix}$, $\begin{pmatrix} z_0^T \\ z_1^T \end{pmatrix}$, resp., $(t \in [0, T])$.

Controllability problem (21)–(23) can be reduced to the following one

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p\mathbf{z} - 2\eta^2(0) v \mathcal{D}_{\eta\theta} \delta, \quad \xi \in \mathbb{R}, \ t \in (0, T), \tag{24}$$

$$\begin{pmatrix} \mathbf{z}(\cdot, 0) \\ \mathbf{z}_t(\cdot, 0) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0^0 \\ \mathbf{z}_1^0 \end{pmatrix} = \mathbf{z}^0 \to \begin{pmatrix} \mathbf{z}(\cdot, T) \\ \mathbf{z}_t(\cdot, T) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0^T \\ \mathbf{z}_1^T \end{pmatrix} = \mathbf{z}^T, \tag{25}$$

Reduced control problem

Put $\widetilde{\mathbb{H}}^m = \{\varphi \in \mathbb{H}^m : \varphi \text{ is odd}\}$, $-2 \le m \le 2$, $\mathbb{I\!H\!I} = \widetilde{\mathbb{H}}^0 \times \widetilde{\mathbb{H}}^{-1}$ with the norm $[\![\cdot]\!]$.

Let $\mathbf{z}(\cdot,t)$, $\mathbf{z}^0$, $\mathbf{z}^T$ be the odd extension w.r.t. ξ for $z(\cdot,t)$, $\begin{pmatrix} z_0^0 \\ z_1^0 \end{pmatrix}$, $\begin{pmatrix} z_0^T \\ z_1^T \end{pmatrix}$, resp., ($t \in [0,T]$).

Controllability problem (21)–(23) can be reduced to the following one

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p\mathbf{z} - 2\eta^2(0) v \mathcal{D}_{\eta\theta}\delta, \quad \xi \in \mathbb{R}, \ t \in (0,T), \tag{24}$$

$$\begin{pmatrix} \mathbf{z}(\cdot,0) \\ \mathbf{z}_t(\cdot,0) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0^0 \\ \mathbf{z}_1^0 \end{pmatrix} = \mathbf{z}^0 \to \begin{pmatrix} \mathbf{z}(\cdot,T) \\ \mathbf{z}_t(\cdot,T) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0^T \\ \mathbf{z}_1^T \end{pmatrix} = \mathbf{z}^T, \tag{25}$$

where $\frac{d^p}{dt^p}\mathbf{z} : [0,T] \to \widetilde{\mathbb{H}}^{-p}$, $p = 0,1,2$, $\mathbf{z}^0, \mathbf{z}^T \in \mathbb{I\!H\!I}$, δ is the Dirac distribution, $\delta = H'$, H is the Heaviside function: $H(\xi) = 1$ if $\xi > 0$, and $H(\xi) = 0$ otherwise.

Reduced control problem

Put $\widetilde{\mathbb{H}}^m = \{\varphi \in \mathbb{H}^m : \varphi \text{ is odd}\}$, $-2 \le m \le 2$, $\mathbb{H} = \widetilde{\mathbb{H}}^0 \times \widetilde{\mathbb{H}}^{-1}$ with the norm $[\![\cdot]\!]$.

Let $\mathbf{z}(\cdot,t)$, $\mathbf{z}^0$, $\mathbf{z}^T$ be the odd extension w.r.t. ξ for $z(\cdot,t)$, $\begin{pmatrix} z_0^0 \\ z_1^0 \end{pmatrix}$, $\begin{pmatrix} z_0^T \\ z_1^T \end{pmatrix}$, resp., $(t \in [0,T])$.

Controllability problem (21)–(23) can be reduced to the following one

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p\mathbf{z} - 2\eta^2(0) v \mathcal{D}_{\eta\theta}\delta, \quad \xi \in \mathbb{R}, \; t \in (0,T), \tag{24}$$

$$\begin{pmatrix} \mathbf{z}(\cdot,0) \\ \mathbf{z}_t(\cdot,0) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0^0 \\ \mathbf{z}_1^0 \end{pmatrix} = \mathbf{z}^0 \to \begin{pmatrix} \mathbf{z}(\cdot,T) \\ \mathbf{z}_t(\cdot,T) \end{pmatrix} = \begin{pmatrix} \mathbf{z}_0^T \\ \mathbf{z}_1^T \end{pmatrix} = \mathbf{z}^T, \tag{25}$$

where $\frac{d^p}{dt^p}\mathbf{z} : [0,T] \to \widetilde{\mathbb{H}}^{-p}$, $p = 0,1,2$, $\mathbf{z}^0, \mathbf{z}^T \in \mathbb{H}$, δ is the Dirac distribution, $\delta = H'$, H is the Heaviside function: $H(\xi) = 1$ if $\xi > 0$, and $H(\xi) = 0$ otherwise.

We call this problem the **main** control problem.

Control system for the wave equation with constant coefficients

Put $\widetilde{H}^m = \{\varphi \in H^m : \varphi \text{ is odd}\}$, $-2 \leq m \leq 2$, $\mathbf{H} = \widetilde{H}^0 \times \widetilde{H}^{-1}$ with the norm $|||\cdot|||$.

Control system for the wave equation with constant coefficients

Put $\widetilde{H}^m = \{\varphi \in H^m : \varphi \text{ is odd}\}$, $-2 \leq m \leq 2$, $\mathbf{H} = \widetilde{H}^0 \times \widetilde{H}^{-1}$ with the norm $|||\cdot|||$.

Consider the **auxiliary** control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u\delta', \quad x \in \mathbb{R},\ t \in (0, T), \tag{26}$$

Control system for the wave equation with constant coefficients

Put $\widetilde{H}^m = \{\varphi \in H^m : \varphi \text{ is odd}\}$, $-2 \le m \le 2$, $\mathbf{H} = \widetilde{H}^0 \times \widetilde{H}^{-1}$ with the norm $|\!|\!|\cdot|\!|\!|$.
Consider the **auxiliary** control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u\delta', \quad x \in \mathbb{R},\ t \in (0, T), \tag{26}$$

$$\begin{pmatrix} \mathbf{w}(\cdot, 0) \\ \mathbf{w}_t(\cdot, 0) \end{pmatrix} = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix} = \mathbf{w}^0 \to \begin{pmatrix} \mathbf{w}(\cdot, T) \\ \mathbf{w}_t(\cdot, T) \end{pmatrix} = \begin{pmatrix} \mathbf{w}_0^T \\ \mathbf{w}_1^T \end{pmatrix} = \mathbf{w}^T, \tag{27}$$

Control system for the wave equation with constant coefficients

Put $\widetilde{H}^m = \{\varphi \in H^m : \varphi \text{ is odd}\}$, $-2 \le m \le 2$, $\mathbf{H} = \widetilde{H}^0 \times \widetilde{H}^{-1}$ with the norm $|\!|\!|\cdot|\!|\!|$.

Consider the **auxiliary** control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u\delta', \quad x \in \mathbb{R},\ t \in (0, T), \tag{26}$$

$$\begin{pmatrix} \mathbf{w}(\cdot, 0) \\ \mathbf{w}_t(\cdot, 0) \end{pmatrix} = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix} = \mathbf{w}^0 \to \begin{pmatrix} \mathbf{w}(\cdot, T) \\ \mathbf{w}_t(\cdot, T) \end{pmatrix} = \begin{pmatrix} \mathbf{w}_0^T \\ \mathbf{w}_1^T \end{pmatrix} = \mathbf{w}^T, \tag{27}$$

where $\frac{d^p}{dt^p}\mathbf{w} : [0, T] \to \widetilde{H}^{-p}$, $p = 0, 1, 2$, $\mathbf{w}^0, \mathbf{w}^T \in \mathbf{H}$, δ is the Dirac distribution with respect to x.

Scheme of study

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p\mathbf{z} - 2\eta^2(0) v \mathcal{D}_{\eta\theta}\delta \text{ in } \widetilde{\mathbb{H}}^{-2}$$

$\mathbf{S}$

$\widetilde{\mathbb{T}} = \mathbf{S}\widetilde{\mathbf{T}}_r$

$$\mathbf{y}_{tt} = \mathbf{y}_{\lambda\lambda} - r\mathbf{y} - 2u\delta' \text{ in } \widetilde{H}^{-2}$$

$\widetilde{\mathbf{T}}_r$

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - q^2\mathbf{w} - 2u\delta' \text{ in } \widetilde{H}^{-2}$$

$$p(\xi) = r\big(\sigma(\xi)\big) + q^2, \quad \xi \in \mathbb{R}.$$

Transformations between solutions to the main and the auxiliary control problems

Theorem

Let $\mathbf{w}$ *be a solution to the auxiliary control problem (i. e., problem* (26), (27)*) for some* $u \in L^\infty(0, T)$ *and* $\mathbf{w}^0 \in \widetilde{H}$*. Let* $\mathbf{z}(\cdot, t) = \widetilde{\mathbb{T}}\mathbf{w}(\cdot, t)$*,* $t \in [0, T]$*.*

Transformations between solutions to the main and the auxiliary control problems

Theorem

Let $\mathbf{w}$ *be a solution to the auxiliary control problem (i. e., problem* (26), (27)*) for some* $u \in L^\infty(0, T)$ *and* $\mathbf{w}^0 \in \widetilde{H}$*. Let* $\mathbf{z}(\cdot, t) = \widetilde{\mathbb{T}}\mathbf{w}(\cdot, t)$*,* $t \in [0, T]$*. Then,* $\mathbf{z}$ *is a solution to the main control problem (i. e., problem* (24), (25)*) with* $\mathbf{z}^0 = \widetilde{\mathbb{T}}\mathbf{w}^0$ *and*

Transformations between solutions to the main and the auxiliary control problems

Theorem

Let $\mathbf{w}$ *be a solution to the auxiliary control problem (i. e., problem* (26), (27)*) for some* $u \in L^\infty(0,T)$ *and* $\mathbf{w}^0 \in \widetilde{H}$. *Let* $\mathbf{z}(\cdot,t) = \widetilde{\mathbb{T}}\mathbf{w}(\cdot,t)$, $t \in [0,T]$. *Then,* $\mathbf{z}$ *is a solution to the main control problem (i. e., problem* (24), (25)*) with* $\mathbf{z}^0 = \widetilde{\mathbb{T}}\mathbf{w}^0$ *and*

$$\eta(0)v(t) = u(t) + \int_0^\infty K(0,\xi)\mathbf{w}(\xi,t)\,d\xi, \qquad t \in [0,T]. \tag{28}$$

Transformations between solutions to the main and the auxiliary control problems

Theorem

Let $\mathbf{w}$ *be a solution to the auxiliary control problem (i. e., problem* (26), (27)*) for some* $u \in L^\infty(0,T)$ *and* $\mathbf{w}^0 \in \widetilde{H}$*. Let* $\mathbf{z}(\cdot,t) = \widetilde{\mathbb{T}}\mathbf{w}(\cdot,t)$*,* $t \in [0,T]$*. Then,* $\mathbf{z}$ *is a solution to the main control problem (i. e., problem* (24), (25)*) with* $\mathbf{z}^0 = \widetilde{\mathbb{T}}\mathbf{w}^0$ *and*

$$\eta(0)v(t) = u(t) + \int_0^\infty K(0,\xi)\mathbf{w}(\xi,t)\,d\xi, \qquad t \in [0,T]. \tag{28}$$

Moreover,

$$\left[\!\left[\!\left[\begin{pmatrix} \mathbf{z}(\cdot,t) \\ \mathbf{z}_t(\cdot,t) \end{pmatrix} \right]\!\right]\!\right]^0 \le C_0 \left|\!\left|\!\left| \begin{pmatrix} \mathbf{w}(\cdot,t) \\ \mathbf{w}_t(\cdot,t) \end{pmatrix} \right|\!\right|\!\right|^0, \qquad t \in [0,T], \tag{29}$$

$$\|v\|_{L^\infty(0,T)} \le Q_0(T)\left(\|u\|_{L^\infty(0,T)} + \left|\!\left|\!\left|\mathbf{w}^0\right|\!\right|\!\right|^0\right), \tag{30}$$

where $C_0 > 0$ *and* $Q_0(T) > 0$*.*

Sketch of proof

Since $\mathbf{z}(\cdot, t) = \widetilde{\mathbb{T}}\mathbf{w}(\cdot, t)$, $t \in [0, T]$, and

Sketch of proof

Since $\mathbf{z}(\cdot, t) = \widetilde{\mathbb{T}}\mathbf{w}(\cdot, t)$, $t \in [0, T]$, and

$\widetilde{\mathbb{T}}$ is an isomorphism of $\widetilde{H}^m$ and $\widetilde{\mathbb{H}}^m$, $-2 \leq m \leq 2$,

Sketch of proof

Since $\mathbf{z}(\cdot,t) = \widetilde{\mathbb{T}}\mathbf{w}(\cdot,t)$, $t \in [0,T]$, and

$\widetilde{\mathbb{T}}$ is an isomorphism of $\widetilde{H}^m$ and $\widetilde{\mathbb{H}}^m$, $-2 \le m \le 2$,

we have

$$\left[\!\left[\begin{pmatrix} \mathbf{z}(\cdot,t) \\ \mathbf{z}_t(\cdot,t) \end{pmatrix} \right]\!\right]^0 \le C_0 \left\| \begin{pmatrix} \mathbf{w}(\cdot,t) \\ \mathbf{w}_t(\cdot,t) \end{pmatrix} \right\|^0, \qquad t \in [0,T],$$

i.e., (29) holds.

Sketch of proof

Now, let us prove (28) and (30).

Sketch of proof

Now, let us prove (28) and (30). We have

$$\mathbf{z}(\xi, t) = \left(\widetilde{\mathbb{T}}\mathbf{w}(\cdot, t)\right)(\xi)$$

Sketch of proof

Now, let us prove (28) and (30). We have

$$\mathbf{z}(\xi,t) = \left(\widetilde{\mathbb{T}}\mathbf{w}(\cdot,t)\right)(\xi)$$

$$= \frac{1}{\eta(\xi)}\left(\mathbf{w}(\lambda,t) + \int_{|\lambda|}^{\infty} K(|\lambda|,x)\mathbf{w}(x,t)\,dx\right)\Bigg|_{\lambda=\sigma(\xi)}, \quad x\in\mathbb{R}, \ \ t\in[0,T].$$

$\sigma(\xi) = \int_0^\xi \frac{d\mu}{\theta^2(\mu)},\ \xi\in\mathbb{R}.$

Sketch of proof

Now, let us prove (28) and (30). We have

$$\mathbf{z}(\xi, t) = \left(\widetilde{\mathbb{T}}\mathbf{w}(\cdot, t)\right)(\xi)$$

$$= \frac{1}{\eta(\xi)} \left(\mathbf{w}(\lambda, t) + \int_{|\lambda|}^{\infty} K(|\lambda|, x)\mathbf{w}(x, t)\, dx \right)\Bigg|_{\lambda=\sigma(\xi)}, \quad x \in \mathbb{R}, \quad t \in [0, T].$$

$\sigma(\xi) = \int_0^{\xi} \frac{d\mu}{\theta^2(\mu)}$, $\xi \in \mathbb{R}$.

Therefore

$$v(t) = \mathbf{z}(+0, t) = \frac{1}{\eta(0)} \left(u(t) + \int_0^{\infty} K(0, x)\mathbf{w}(x, t)\, dx \right), \quad t \in [0, T].$$

$u(t) = \mathbf{w}(+0, t)$, $t \in [0, T]$,

i.e., (28) is true.

Sketch of proof

Therefore,

$$|v(t)| \leq \frac{1}{\eta(0)} \left(|u(t)| + \|K(0,\cdot)\|^0 \, \|\mathbf{w}(\cdot,t)\|^0 \right), \ \ t \in [0, T]$$

Sketch of proof

Therefore,

$$|v(t)| \le \frac{1}{\eta(0)} \left(|u(t)| + \|K(0,\cdot)\|^0 \|\mathbf{w}(\cdot,t)\|^0 \right), \ t \in [0, T]$$

We have

$$\left(\|K(0,\cdot)\|^0 \right)^2 \le M_0 \int_0^\infty \left(\sigma_0\left(\frac{x}{2}\right)\right)^2 dx \le 2M_0\sigma_0(0) \int_0^\infty x r(x)\, dx = C$$

$|K(y)| \le M_0\sigma_0\left(\frac{y_1+y_2}{2}\right)$, $y_2 \ge y_1 \ge 0$, $\sigma_0(x) = \int_x^\infty |r(\xi)|\, d\xi$, $x > 0$.

Sketch of proof

Therefore,

$$|v(t)| \le \frac{1}{\eta(0)} \left(|u(t)| + \|K(0,\cdot)\|^0 \, \|\mathbf{w}(\cdot,t)\|^0 \right), \ \ t \in [0, T]$$

We have

$$\left(\|K(0,\cdot)\|^0 \right)^2 \le M_0 \int_0^\infty \left(\sigma_0 \left(\frac{x}{2} \right) \right)^2 dx \le 2M_0\sigma_0(0) \int_0^\infty x r(x)\, dx = C$$

$|K(y)| \le M_0\sigma_0 \left(\frac{y_1+y_2}{2} \right)$, $y_2 \ge y_1 \ge 0$, $\sigma_0(x) = \int_x^\infty |r(\xi)|\, d\xi$, $x > 0$.

Hence,

$$\|v\|_{L^2(0,T)} \le \frac{1}{\eta(0)} \left(\|u\|_{L^2(0,T)} + CQ(t) \left(\left\| \mathbf{w}^0 \right\|^0 + \|u\|_{L^\infty(0,T)} \right) \right)$$

$\|\mathbf{w}(\cdot,t)\|^0 \le Q(T) \left(\left\| \mathbf{w}^0 \right\|^0 + \|u\|_{L^\infty(0,T)} \right)$, $t \in [0, T]$,

i.e., (30) holds. □

Transformations between solutions to the main and the auxiliary control problems

Theorem

Let $\mathbf{z}$ *be a solution to the main control problem (i. e., problem* (24), (25)*) for some* $v \in L^\infty(0, T)$ *and* $\mathbf{z}^0 \in \mathbb{HI}$*. Let* $\mathbf{w}(\cdot, t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot, t)$*,* $t \in [0, T]$*.*

Transformations between solutions to the main and the auxiliary control problems

Theorem

Let $\mathbf{z}$ be a solution to the main control problem (i. e., problem (24), (25)*) for some $v \in L^\infty(0, T)$ and $\mathbf{z}^0 \in \mathbb{H}$. Let $\mathbf{w}(\cdot, t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot, t)$, $t \in [0, T]$. Then, $\mathbf{w}$ is a solution to the auxiliary control problem (i. e., problem* (26), (27)*) with $\mathbf{w}^0 = \widetilde{\mathbb{T}}^{-1}\mathbf{z}^0$ and*

Transformations between solutions to the main and the auxiliary control problems

Theorem

Let $\mathbf{z}$ *be a solution to the main control problem (i. e., problem* (24), (25)*) for some* $v \in L^\infty(0, T)$ *and* $\mathbf{z}^0 \in \mathbb{H}$*. Let* $\mathbf{w}(\cdot, t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot, t)$, $t \in [0, T]$*. Then,* $\mathbf{w}$ *is a solution to the auxiliary control problem (i. e., problem* (26), (27)*) with* $\mathbf{w}^0 = \widetilde{\mathbb{T}}^{-1}\mathbf{z}^0$ *and*

$$u(t) = \eta(0)v(t) + \int_0^\infty L(0, x)\mathbf{S}^{-1}\mathbf{z}(x, t)\, dx, \qquad t \in [0, T]. \tag{31}$$

Transformations between solutions to the main and the auxiliary control problems

Theorem

Let $\mathbf{z}$ *be a solution to the main control problem (i. e., problem* (24), (25)*) for some* $v \in L^\infty(0,T)$ *and* $\mathbf{z}^0 \in \mathbb{H}$*. Let* $\mathbf{w}(\cdot,t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot,t)$*,* $t \in [0,T]$*. Then,* $\mathbf{w}$ *is a solution to the auxiliary control problem (i. e., problem* (26), (27)*) with* $\mathbf{w}^0 = \widetilde{\mathbb{T}}^{-1}\mathbf{z}^0$ *and*

$$u(t) = \eta(0)v(t) + \int_0^\infty L(0,x)\mathbf{S}^{-1}\mathbf{z}(x,t)\,dx, \qquad t \in [0,T]. \tag{31}$$

Moreover,

$$\left\| \begin{pmatrix} \mathbf{w}(\cdot,t) \\ \mathbf{w}_t(\cdot,t) \end{pmatrix} \right\|^0 \le C_1 \left[\!\left[\begin{pmatrix} \mathbf{z}(\cdot,t) \\ \mathbf{z}_t(\cdot,t) \end{pmatrix} \right]\!\right]^0, \qquad t \in [0,T], \tag{32}$$

$$\|u\|_{L^\infty(0,T)} \le Q_1(T)\left(\|v\|_{L^\infty(0,T)} + [\![\mathbf{z}^0]\!]^0\right), \tag{33}$$

where $C_1 > 0$ *and* $Q_1(T) > 0$*.*

Sketch of proof

The proof of conditions (31) and (32) is similar to the proof of the appropriate conditions of the previous theorem.

Sketch of proof

The proof of conditions (31) and (32) is similar to the proof of the appropriate conditions of the previous theorem.
Let us prove (33).

Sketch of proof

The proof of conditions (31) and (32) is similar to the proof of the appropriate conditions of the previous theorem.

Let us prove (33).

Since $\mathbf{w}(\cdot, t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot, t) = \widetilde{\mathbf{T}}_r^{-1}\mathbf{S}^{-1}\mathbf{z}(\cdot, t)$, $t \in [0, T]$, we have

$$u(t) = \eta(0)v(t) - \int_0^\infty K(0, x)\mathbf{w}(\lambda, t)\, dx$$

$\mathbf{w}$ depends on w^0 and u.

Sketch of proof

The proof of conditions (31) and (32) is similar to the proof of the appropriate conditions of the previous theorem.

Let us prove (33).

Since $\mathbf{w}(\cdot, t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot, t) = \widetilde{\mathbf{T}}_r^{-1}\mathbf{S}^{-1}\mathbf{z}(\cdot, t)$, $t \in [0, T]$, we have

$$u(t) = \eta(0)v(t) - \int_0^\infty K(0, x)\mathbf{w}(\lambda, t)\, dx$$

w depends on w^0 and u.

Therefore,

$$u(t) = g(t) + \int_0^t P(t-\mu)u(\mu)\, d\mu, \qquad t \in [0, T],$$

Sketch of proof

The proof of conditions (31) and (32) is similar to the proof of the appropriate conditions of the previous theorem.

Let us prove (33).

Since $\mathbf{w}(\cdot, t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot, t) = \widetilde{\mathbf{T}}_r^{-1}\mathbf{S}^{-1}\mathbf{z}(\cdot, t)$, $t \in [0, T]$, we have

$$u(t) = \eta(0)v(t) - \int_0^\infty K(0, x)\mathbf{w}(\lambda, t)\, dx$$

$\mathbf{w}$ depends on w^0 and u.

Therefore,

$$u(t) = g(t) + \int_0^t P(t-\mu)u(\mu)\, d\mu, \qquad t \in [0, T],$$

where g depends on v, $\mathbf{w}^0$, K, and P depends on K,

$$g \in L^\infty(0, T) \qquad \text{and} \qquad P \in L^\infty(0, T).$$

Sketch of proof

Thus, u is determined by the integral equation

$$u(t) = g(t) + \int_0^t P(t-\mu)u(\mu)\,d\mu, \qquad t \in [0, T]. \tag{34}$$

Sketch of proof

Thus, u is determined by the integral equation

$$u(t) = g(t) + \int_0^t P(t-\mu)u(\mu)\,d\mu, \qquad t \in [0, T]. \tag{34}$$

It follows from

Theorem (Gronwall). Let $y \in L^1(0, T)$, $y(t) \geq 0$, $t \in (0, T)$, and $y(t) \leq C_1 + C_2 \int_0^t y(\lambda)\,d\lambda$, $t \in (0, T)$, for some constants $C_1, C_2 > 0$. Then $y(t) \leq C_1 e^{tC_2}$, $t \in (0, T)$.

Sketch of proof

Thus, u is determined by the integral equation

$$u(t) = g(t) + \int_0^t P(t-\mu)u(\mu)\,d\mu, \qquad t \in [0, T]. \tag{34}$$

It follows from

Theorem (Gronwall). Let $y \in L^1(0, T)$, $y(t) \geq 0$, $t \in (0, T)$, and $y(t) \leq C_1 + C_2 \int_0^t y(\lambda)\,d\lambda$, $t \in (0, T)$, for some constants $C_1, C_2 > 0$. Then $y(t) \leq C_1 e^{tC_2}$, $t \in (0, T)$.

that the equation

$$u(t) = \int_0^t P(t-\mu)u(\mu)\,d\mu, \qquad t \in [0, T],$$

has only trivial solution in $L^2(0, T)$.

Sketch of proof

Thus, u is determined by the integral equation

$$u(t) = g(t) + \int_0^t P(t-\mu)u(\mu)\,d\mu, \qquad t \in [0, T]. \tag{34}$$

It follows from

Theorem (Gronwall). Let $y \in L^1(0, T)$, $y(t) \geq 0$, $t \in (0, T)$, and $y(t) \leq C_1 + C_2 \int_0^t y(\lambda)\,d\lambda$, $t \in (0, T)$, for some constants $C_1, C_2 > 0$. Then $y(t) \leq C_1 e^{tC_2}$, $t \in (0, T)$.

that the equation

$$u(t) = \int_0^t P(t-\mu)u(\mu)\,d\mu, \qquad t \in [0, T],$$

has only trivial solution in $L^2(0, T)$.
By using the Fredholm alternative, we see that equation (34) has the unique solution in $L^2(0, T)$.

Sketch of proof

It follows from (34) that

$$|u(t)| \le \|g\|_{L^\infty(0,T)} + \|P\|_{L^\infty(0,T)} \int_0^t |u(\mu)|\, d\mu, \qquad t \in [0, T].$$

Sketch of proof

It follows from (34) that

$$|u(t)| \leq \|g\|_{L^\infty(0,T)} + \|P\|_{L^\infty(0,T)} \int_0^t |u(\mu)|\, d\mu, \qquad t \in [0, T].$$

Applying again

Theorem (Gronwall). Let $y \in L^1(0, T)$, $y(t) \geq 0$, $t \in (0, T)$, and $y(t) \leq C_1 + C_2 \int_0^t y(\lambda)\, d\lambda$, $t \in (0, T)$, for some constants $C_1, C_2 > 0$. Then $y(t) \leq C_1 e^{tC_2}$, $t \in (0, T)$.

Sketch of proof

It follows from (34) that

$$|u(t)| \le \|g\|_{L^\infty(0,T)} + \|P\|_{L^\infty(0,T)} \int_0^t |u(\mu)|\,d\mu, \qquad t \in [0,T].$$

Applying again

Theorem (Gronwall). Let $y \in L^1(0,T)$, $y(t) \ge 0$, $t \in (0,T)$, and $y(t) \le C_1 + C_2 \int_0^t y(\lambda)\,d\lambda$, $t \in (0,T)$, for some constants $C_1, C_2 > 0$. Then $y(t) \le C_1 e^{tC_2}$, $t \in (0,T)$.

we obtain

$$|u(t)| \le \|g\|_{L^\infty(0,T)}\, e^{t\|P\|_{L^\infty(0,T)}}, \qquad t \in [0,T],$$

$\|g\|_{L^\infty(0,t)}$ depends on $\left|\!\left|\!\left| w^0 \right|\!\right|\!\right|^0$ and $\|v\|_{L^\infty(0,t)}$.

Sketch of proof

It follows from (34) that

$$|u(t)| \leq \|g\|_{L^\infty(0,T)} + \|P\|_{L^\infty(0,T)} \int_0^t |u(\mu)|\, d\mu, \qquad t \in [0,T].$$

Applying again

Theorem (Gronwall). Let $y \in L^1(0,T)$, $y(t) \geq 0$, $t \in (0,T)$, and $y(t) \leq C_1 + C_2 \int_0^t y(\lambda)\, d\lambda$, $t \in (0,T)$, for some constants $C_1, C_2 > 0$. Then $y(t) \leq C_1 e^{tC_2}$, $t \in (0,T)$.

we obtain

$$|u(t)| \leq \|g\|_{L^\infty(0,T)}\, e^{t\|P\|_{L^\infty(0,T)}}, \qquad t \in [0,T],$$

$\|g\|_{L^\infty(0,t)}$ depends on $\left\| w^0 \right\|^0$ and $\|v\|_{L^\infty(0,t)}$.

Therefore,

$$\|u\|_{L^\infty(0,T)} \leq Q_1(T) \left(\|v\|_{L^\infty(0,T)} + [\![\mathbf{z}^0]\!]^0 \right)$$

for some $Q_1(T) > 0$. □

Uniqueness and well-posedness of the main control problem

Remark *It is well known that the solution to the auxiliary control problem (i. e., problem* (26), (27)*) is unique. Therefore, the last two theorems yield uniqueness of solution to the main control problem (i. e., problem* (24), (25)*).*

Uniqueness and well-posedness of the main control problem

Remark *It is well known that the solution to the auxiliary control problem (i. e., problem* (26), (27)*) is unique. Therefore, the last two theorems yield uniqueness of solution to the main control problem (i. e., problem* (24), (25)*).*

Remark *It follows from the last two theorems that*

$$\left[\!\left[\begin{pmatrix} \mathbf{z}(\cdot,t) \\ \mathbf{z}_t(\cdot,t) \end{pmatrix} \right]\!\right]^0 \le Q_2(T)\left([\![\mathbf{z}^0]\!]^0 + \|v\|_{L^\infty(0,T)} \right), \qquad t \in [0,T],$$

where $Q_2(T) > 0$*. Therefore, the main control problem (i. e., problem* (24), (25)*) is well posed.*

Necessary and sufficient conditions of approximate null-controllability for the main control problem at a free time

Thus we obtain the following theorem

Theorem

Let $q > 0$. Each state $\mathbf{z}^0 \in \mathbb{H}$ of the main control problem (i. e., problem (24), (25)*) is approximately null-controllable at a* ***free*** *time.*

Necessary and sufficient conditions of approximate null-controllability for the main control problem at a free time

Thus we obtain the following theorem

Theorem

Let $q > 0$. Each state $\mathbf{z}^0 \in \mathbb{H}$ of the main control problem (i. e., problem (24), (25)*) is approximately null-controllable at a* **free** *time.*

Theorem

Let $q = 0$. A state $\mathbf{z}^0 \in \mathbb{H}$ of the main control problem (i. e., problem (24), (25)*) is approximately null-controllable at a* **free** *time iff*

$$\mathbf{z}_1^0 - \widetilde{\mathbb{T}} \left(\operatorname{sgn}(\cdot)\, \widetilde{\mathbb{T}}^{-1} \mathbf{z}_0^0 \right)' = 0. \tag{35}$$

Example

Consider the following control problem

$$
\begin{aligned}
&z_{tt} = (1+\xi)\left((1+\xi)\,z_\xi\right)_\xi - \frac{4+3\xi}{4(1+\xi)}z, \ \ \xi > 0, \ \ t \in (0,T)\\
&z(0,t) = v(t), \qquad t \in (0,T),\\
&z(\xi,0) = z_0^0(\xi) = 2I_2\left(\frac{2}{\sqrt{1+\xi}}\right), \qquad \xi > 0,\\
&z_t(\xi,0) = z_1^0(\xi) = -I_2\left(\frac{2}{\sqrt{1+\xi}}\right), \qquad \xi > 0,
\end{aligned}
$$

where $v \in L^\infty(0,T)$ is a control.

Example

Let us construct the spaces $\mathbb{H}^m$, $m = \overline{2, -2}$, where this problem is considered and obtain the reduced control problem.

Example

Let us construct the spaces $\mathbb{H}^m$, $m = \overline{2, -2}$, where this problem is considered and obtain the reduced control problem.
We have $\rho(x) = \frac{1}{1+|\xi|}$, $k(x) = 1 + |\xi|$.

Example

Let us construct the spaces $\mathbb{H}^m$, $m = \overline{2, -2}$, where this problem is considered and obtain the reduced control problem.
We have $\rho(x) = \frac{1}{1+|\xi|}$, $k(x) = 1 + |\xi|$. Then,

$$\eta(\xi) = (k(\xi)\rho(\xi))^{1/4} = 1,$$

Example

Let us construct the spaces $\mathbb{H}^m$, $m = \overline{2, -2}$, where this problem is considered and obtain the reduced control problem.
We have $\rho(x) = \frac{1}{1+|\xi|}$, $k(x) = 1 + |\xi|$. Then,

$$\eta(\xi) = (k(\xi)\rho(\xi))^{1/4} = 1,$$

$$\theta(\xi) = (k(\xi)/\rho(\xi))^{1/4} = \sqrt{1 + |\xi|},$$

Example

Let us construct the spaces $\mathbb{H}^m$, $m = \overline{2, -2}$, where this problem is considered and obtain the reduced control problem.
We have $\rho(x) = \frac{1}{1+|\xi|}$, $k(x) = 1 + |\xi|$. Then,

$$\eta(\xi) = (k(\xi)\rho(\xi))^{1/4} = 1,$$

$$\theta(\xi) = (k(\xi)/\rho(\xi))^{1/4} = \sqrt{1 + |\xi|},$$

$$\sigma(\xi) = \operatorname{sgn}\xi \, \ln(1 + |\xi|),$$

Example

Let us construct the spaces $\mathbb{H}^m$, $m = \overline{2, -2}$, where this problem is considered and obtain the reduced control problem.
We have $\rho(x) = \frac{1}{1+|\xi|}$, $k(x) = 1 + |\xi|$. Then,

$$\eta(\xi) = (k(\xi)\rho(\xi))^{1/4} = 1,$$

$$\theta(\xi) = (k(\xi)/\rho(\xi))^{1/4} = \sqrt{1 + |\xi|},$$

$$\sigma(\xi) = \operatorname{sgn} \xi \, \ln(1 + |\xi|),$$

$$\mathcal{D}_{\eta\theta} = \theta^2(\xi) \left(\frac{d}{d\xi} + \frac{\eta'}{\eta} \right) = (1 + |\xi|) \frac{d}{d\xi}.$$

Example

Let us construct the spaces $\mathbb{H}^m$, $m = \overline{2, -2}$, where this problem is considered and obtain the reduced control problem.
We have $\rho(x) = \frac{1}{1+|\xi|}$, $k(x) = 1 + |\xi|$. Then,

$$\eta(\xi) = (k(\xi)\rho(\xi))^{1/4} = 1,$$

$$\theta(\xi) = (k(\xi)/\rho(\xi))^{1/4} = \sqrt{1 + |\xi|},$$

$$\sigma(\xi) = \operatorname{sgn} \xi \, \ln(1 + |\xi|),$$

$$\mathcal{D}_{\eta\theta} = \theta^2(\xi) \left(\frac{d}{d\xi} + \frac{\eta'}{\eta} \right) = (1 + |\xi|) \frac{d}{d\xi}.$$

$$(\mathbf{S}\psi)(\xi) = \psi(\sigma(\xi)),\ \psi \in H^m, \quad \langle\langle \mathbf{S}g, \varphi \rangle\rangle = \langle g, \mathbf{S}^{-1}\varphi \rangle,\ g \in H^{-m},\ \varphi \in \mathbb{H}^m.$$

Example

We have

$$\begin{aligned}\mathcal{D}_{\eta\theta}\varphi =&(1+|\xi|)\varphi',\\ \mathcal{D}_{\eta\theta}^2\varphi =&(1+|\xi|)\frac{d}{d\xi}\left((1+|\xi|)\varphi\right) = (1+|\xi|)\varphi' \operatorname{sgn}\xi + (1+|\xi|)^2\varphi''\end{aligned}$$

Example

We have

$$\mathcal{D}_{\eta\theta}\varphi = (1+|\xi|)\varphi',$$
$$\mathcal{D}_{\eta\theta}^2\varphi = (1+|\xi|)\frac{d}{d\xi}\left((1+|\xi|)\varphi\right) = (1+|\xi|)\varphi' \operatorname{sgn}\xi + (1+|\xi|)^2\varphi''$$

Hence

$$\varphi \in \mathbb{H}^m \Leftrightarrow \mathcal{D}_{\eta\theta}^m\varphi \in L^2_{\eta\theta}(\mathbb{R}) \Leftrightarrow (1+|\xi|)^m\varphi^{(m)} \in L^2_{\eta\theta}(\mathbb{R}), \;\; m = 0, 1, 2,$$
$$\mathbb{H}^{-m} = \left(\mathbb{H}^m\right)^*, \;\; m = 0, 1, 2,$$
$$\langle\langle f, \varphi\rangle\rangle = \left\langle \mathbf{S}^{-1} f, \mathbf{S}^{-1}\varphi \right\rangle.$$

where $L^2_{\eta\theta}(\mathbb{R})$ is the space of functions square-integrable on $\mathbb{R}$ with the weight η^2/θ^2.

Example

Let $\mathbf{z}(\cdot, t)$, $\mathbf{z}_0^0$, $\mathbf{z}_1^0$ be the odd extension w.r.t. ξ for $z(\cdot, t)$, z_0^0, z_1^0, resp., ($t \in [0, T]$).

Example

Let $\mathbf{z}(\cdot, t)$, $\mathbf{z}_0^0$, $\mathbf{z}_1^0$ be the odd extension w.r.t. ξ for $z(\cdot, t)$, z_0^0, z_1^0, resp., ($t \in [0, T]$).
The control problem can be reduced to the following one

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p(\xi)\mathbf{z} - 2\eta^2(0) v(t) \mathcal{D}_{\eta\theta} \delta(\xi), \quad \xi \in \mathbb{R},\ t \in (0, T),$$

Example

Let $\mathbf{z}(\cdot, t)$, $\mathbf{z}_0^0$, $\mathbf{z}_1^0$ be the odd extension w.r.t. ξ for $z(\cdot, t)$, z_0^0, z_1^0, resp., ($t \in [0, T]$).
The control problem can be reduced to the following one

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p(\xi)\mathbf{z} - 2\eta^2(0)v(t)\mathcal{D}_{\eta\theta}\delta(\xi), \quad \xi \in \mathbb{R}, \ t \in (0, T),$$

$$\mathbf{z}(\cdot, 0) = \mathbf{z}_0^0, \qquad \mathbf{z}_t(\cdot, 0) = \mathbf{z}_1^0,$$

Example

Let $\mathbf{z}(\cdot, t)$, $\mathbf{z}_0^0$, $\mathbf{z}_1^0$ be the odd extension w.r.t. ξ for $z(\cdot, t)$, z_0^0, z_1^0, resp., $(t \in [0, T])$.
The control problem can be reduced to the following one

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p(\xi)\mathbf{z} - 2\eta^2(0)v(t)\mathcal{D}_{\eta\theta}\delta(\xi), \quad \xi \in \mathbb{R}, \ t \in (0, T),$$

$$\mathbf{z}(\cdot, 0) = \mathbf{z}_0^0, \qquad \mathbf{z}_t(\cdot, 0) = \mathbf{z}_1^0,$$

where $\frac{d^p}{dt^p}\mathbf{z} : [0, T] \to \widetilde{\mathbb{H}}^{-p}$, $p = 0, 1, 2$, $\mathbf{z}_0^0 \in \widetilde{\mathbb{H}}^0$, $\mathbf{z}_1^0 \in \widetilde{\mathbb{H}}^{-1}$,

$$p(\xi) = \frac{4 + 3|\xi|}{4(1 + |\xi|)}.$$

Example

Let $\mathbf{z}(\cdot, t)$, $\mathbf{z}_0^0$, $\mathbf{z}_1^0$ be the odd extension w.r.t. ξ for $z(\cdot, t)$, z_0^0, z_1^0, resp., $(t \in [0, T])$.
The control problem can be reduced to the following one

$$\mathbf{z}_{tt} = \mathcal{D}_{\eta\theta}^2 \mathbf{z} + p(\xi)\mathbf{z} - 2\eta^2(0)v(t)\mathcal{D}_{\eta\theta}\delta(\xi), \quad \xi \in \mathbb{R}, \ t \in (0, T),$$

$$\mathbf{z}(\cdot, 0) = \mathbf{z}_0^0, \qquad \mathbf{z}_t(\cdot, 0) = \mathbf{z}_1^0,$$

where $\frac{d^p}{dt^p}\mathbf{z} : [0, T] \to \widetilde{\mathbb{H}}^{-p}$, $p = 0, 1, 2$, $\mathbf{z}_0^0 \in \widetilde{\mathbb{H}}^0$, $\mathbf{z}_1^0 \in \widetilde{\mathbb{H}}^{-1}$,

$$p(\xi) = \frac{4 + 3|\xi|}{4(1 + |\xi|)}.$$

We call this problem the **main** control problem.

Example

We have

$$\left(p\circ\sigma^{-1}\right)(\lambda)=\frac{3}{4}+e^{-|\lambda|},\qquad \lambda\in\mathbb{R},$$

$$p(\xi)=\frac{4+3|\xi|}{4(1+|\xi|)},\ \sigma^{-1}(\lambda)=\left(e^{-|\lambda|}-1\right)\operatorname{sgn}\lambda.$$

Example

We have

$$\left(p \circ \sigma^{-1}\right)(\lambda) = \frac{3}{4} + e^{-|\lambda|}, \qquad \lambda \in \mathbb{R},$$

$$p(\xi) = \frac{4 + 3|\xi|}{4(1 + |\xi|)}, \ \sigma^{-1}(\lambda) = \left(e^{-|\lambda|} - 1\right) \operatorname{sgn} \lambda.$$

Hence $q = \frac{\sqrt{3}}{2} > 0$, $r(\lambda) = e^{-|\lambda|}$, $\lambda \in \mathbb{R}$.

$$\int_0^\infty \lambda r(\lambda)\, d\lambda < \infty.$$

Example

Denote $\mathbf{w}(\cdot,t)=\widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot,t),\ t\in[0,T]$, $\mathbf{w}_0^0=\widetilde{\mathbb{T}}^{-1}\mathbf{z}_0^0$, $\mathbf{w}_1^0=\widetilde{\mathbb{T}}^{-1}\mathbf{z}_1^0$.

Example

Denote $\mathbf{w}(\cdot,t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot,t)$, $t \in [0,T]$, $\mathbf{w}_0^0 = \widetilde{\mathbb{T}}^{-1}\mathbf{z}_0^0$, $\mathbf{w}_1^0 = \widetilde{\mathbb{T}}^{-1}\mathbf{z}_1^0$.
Then we obtain the auxiliary control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - \frac{3}{4}\mathbf{w} - 2u\delta', \quad x \in \mathbb{R},\ t \in (0,T),$$

Example

Denote $\mathbf{w}(\cdot, t) = \widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot, t)$, $t \in [0, T]$, $\mathbf{w}_0^0 = \widetilde{\mathbb{T}}^{-1}\mathbf{z}_0^0$, $\mathbf{w}_1^0 = \widetilde{\mathbb{T}}^{-1}\mathbf{z}_1^0$.
Then we obtain the auxiliary control problem

$$\mathbf{w}_{tt} = \mathbf{w}_{xx} - \frac{3}{4}\mathbf{w} - 2u\delta', \quad x \in \mathbb{R},\ t \in (0, T),$$

$$\mathbf{w}(\cdot, 0) = \mathbf{w}_0^0, \qquad \mathbf{w}_t(\cdot, 0) = \mathbf{w}_1^0,$$

Example

Denote $\mathbf{w}(\cdot,t)=\widetilde{\mathbb{T}}^{-1}\mathbf{z}(\cdot,t)$, $t\in[0,T]$, $\mathbf{w}_0^0=\widetilde{\mathbb{T}}^{-1}\mathbf{z}_0^0$, $\mathbf{w}_1^0=\widetilde{\mathbb{T}}^{-1}\mathbf{z}_1^0$.
Then we obtain the auxiliary control problem

$$\mathbf{w}_{tt}=\mathbf{w}_{xx}-\frac{3}{4}\mathbf{w}-2u\delta', \quad x\in\mathbb{R},\ t\in(0,T),$$

$$\mathbf{w}(\cdot,0)=\mathbf{w}_0^0, \qquad \mathbf{w}_t(\cdot,0)=\mathbf{w}_1^0,$$

where $\frac{d^p}{dt^p}\mathbf{w}:[0,T]\to\widetilde{H}^{-p}$, $p=0,1,2$, $\mathbf{w}_0^0\in\widetilde{H}^0$, $\mathbf{w}_1^0\in\widetilde{H}^{-1}$,

$$u(t)=v(t)+\int_0^\infty L(0,\lambda)\mathbf{z}(e^{-\lambda}-1,t)\,d\lambda.$$

$$L(y)=\frac{\partial}{\partial y_1}J_0\left(2\sqrt{e^{-\frac{y_2}{2}}\left(e^{-\frac{y_1}{2}}-e^{-\frac{y_2}{2}}\right)}\right),\ y_2\geq y_1\geq 0.$$

Example

Calculating $\mathbf{w}_0^0$ and $\mathbf{w}_1^0$, we obtain

$$\mathbf{w}_0^0(x) = e^{-|x|}\operatorname{sgn} x \quad \text{and} \quad \mathbf{w}_1^0(x) = -\frac{1}{2}e^{-|x|}\operatorname{sgn} x, \qquad x \in \mathbb{R}.$$

Example

Calculating $\mathbf{w}_0^0$ and $\mathbf{w}_1^0$, we obtain

$$\mathbf{w}_0^0(x) = e^{-|x|}\operatorname{sgn} x \quad \text{and} \quad \mathbf{w}_1^0(x) = -\frac{1}{2}e^{-|x|}\operatorname{sgn} x, \qquad x \in \mathbb{R}.$$

For $u_n(t) = e^{-t/2}$, $t \in [0, T_n]$,

Example

Calculating $\mathbf{w}_0^0$ and $\mathbf{w}_1^0$, we obtain

$$\mathbf{w}_0^0(x) = e^{-|x|}\operatorname{sgn} x \quad \text{and} \quad \mathbf{w}_1^0(x) = -\frac{1}{2}e^{-|x|}\operatorname{sgn} x, \qquad x \in \mathbb{R}.$$

For $u_n(t) = e^{-t/2}$, $t \in [0, T_n]$, we obtain that

$$\mathbf{w}^n(x,t) = e^{-t/2}e^{-|x|}\operatorname{sgn} x, \quad x \in \mathbb{R},\ t \in [0, T_n],$$

is the solution to the auxiliary control problem with $u = u_n$ and $T = T_n$.

Example

Calculating $\mathbf{w}_0^0$ and $\mathbf{w}_1^0$, we obtain

$$\mathbf{w}_0^0(x) = e^{-|x|}\operatorname{sgn} x \quad \text{and} \quad \mathbf{w}_1^0(x) = -\frac{1}{2}e^{-|x|}\operatorname{sgn} x, \qquad x \in \mathbb{R}.$$

For $u_n(t) = e^{-t/2}$, $t \in [0, T_n]$, we obtain that

$$\mathbf{w}^n(x,t) = e^{-t/2}e^{-|x|}\operatorname{sgn} x, \quad x \in \mathbb{R},\ t \in [0, T_n],$$

is the solution to the auxiliary control problem with $u = u_n$ and $T = T_n$. Thus the state $\mathbf{w}^0 = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix}$ is approximately null-controllable at a free time.

Example

Calculating $\mathbf{w}_0^0$ and $\mathbf{w}_1^0$, we obtain

$$\mathbf{w}_0^0(x) = e^{-|x|} \operatorname{sgn} x \quad \text{and} \quad \mathbf{w}_1^0(x) = -\frac{1}{2} e^{-|x|} \operatorname{sgn} x, \qquad x \in \mathbb{R}.$$

For $u_n(t) = e^{-t/2}$, $t \in [0, T_n]$, we obtain that

$$\mathbf{w}^n(x,t) = e^{-t/2} e^{-|x|} \operatorname{sgn} x, \quad x \in \mathbb{R},\ t \in [0, T_n],$$

is the solution to the auxiliary control problem with $u = u_n$ and $T = T_n$. Thus the state $\mathbf{w}^0 = \begin{pmatrix} \mathbf{w}_0^0 \\ \mathbf{w}_1^0 \end{pmatrix}$ is approximately null-controllable at a free time.

Moreover, the pairs (T_n, u_n) ($T_n \to \infty$ as $n \to \infty$), solve the approximate null-controllability problem at a free time.

Example

Since $\mathbf{z}^n(\cdot, t) = \widetilde{\mathbb{T}}\mathbf{w}^n(\cdot, t)$, $t \in [0, T_n]$, we have

$$\mathbf{z}^n(\xi, t) = 2e^{-t/2} I_2\left(\frac{2}{\sqrt{1+|\xi|}}\right) \operatorname{sgn}\xi, \quad \xi \in \mathbb{R},\ t \in [0, T_n],$$

Example

Since $\mathbf{z}^n(\cdot, t) = \widetilde{\mathbb{T}}\mathbf{w}^n(\cdot, t)$, $t \in [0, T_n]$, we have

$$\mathbf{z}^n(\xi, t) = 2e^{-t/2} I_2\left(\frac{2}{\sqrt{1+|\xi|}}\right) \operatorname{sgn}\xi, \quad \xi \in \mathbb{R},\ t \in [0, T_n],$$

and $\mathbf{z}^n$ is the solution to the main control problem with $T = T_n$ and

$$v(t) = v_n(t) = u_n(t) + \int_0^\infty K(0, x)\mathbf{w}^n(x, t)\, dx = 2I_2(2)e^{-t/2}, \quad t \in [0, T_n].$$

$$K(y) = \frac{\partial}{\partial y_2} I_0\left(2\sqrt{e^{-\frac{y_1}{2}}\left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}}\right)}\right),\ y_2 \geq y_1 \geq 0.$$

Example

Since $\mathbf{z}^n(\cdot, t) = \widetilde{\mathbb{T}}\mathbf{w}^n(\cdot, t)$, $t \in [0, T_n]$, we have

$$\mathbf{z}^n(\xi, t) = 2e^{-t/2} I_2\left(\frac{2}{\sqrt{1+|\xi|}}\right) \operatorname{sgn}\xi, \quad \xi \in \mathbb{R},\ t \in [0, T_n],$$

and $\mathbf{z}^n$ is the solution to the main control problem with $T = T_n$ and

$$v(t) = v_n(t) = u_n(t) + \int_0^\infty K(0, x)\mathbf{w}^n(x, t)\, dx = 2I_2(2)e^{-t/2}, \quad t \in [0, T_n].$$

$$K(y) = \frac{\partial}{\partial y_2} I_0\left(2\sqrt{e^{-\frac{y_1}{2}}\left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}}\right)}\right),\ y_2 \geq y_1 \geq 0.$$

Thus the state $\mathbf{z}^0 = \begin{pmatrix} \mathbf{z}_0^0 \\ \mathbf{z}_1^0 \end{pmatrix}$ is approximately null-controllable at a free time.

Example

Since $\mathbf{z}^n(\cdot, t) = \widetilde{\mathbb{T}}\mathbf{w}^n(\cdot, t)$, $t \in [0, T_n]$, we have

$$\mathbf{z}^n(\xi, t) = 2e^{-t/2} I_2\left(\frac{2}{\sqrt{1+|\xi|}}\right) \operatorname{sgn}\xi, \quad \xi \in \mathbb{R}, \ t \in [0, T_n],$$

and $\mathbf{z}^n$ is the solution to the main control problem with $T = T_n$ and

$$v(t) = v_n(t) = u_n(t) + \int_0^\infty K(0, x)\mathbf{w}^n(x, t)\, dx = 2I_2(2)e^{-t/2}, \quad t \in [0, T_n].$$

$$K(y) = \frac{\partial}{\partial y_2} I_0\left(2\sqrt{e^{-\frac{y_1}{2}}\left(e^{-\frac{y_1}{2}} - e^{-\frac{y_2}{2}}\right)}\right), \ y_2 \geq y_1 \geq 0.$$

Thus the state $\mathbf{z}^0 = \begin{pmatrix} \mathbf{z}_0^0 \\ \mathbf{z}_1^0 \end{pmatrix}$ is approximately null-controllable at a free time.

Moreover, the pairs (T_n, v_n) ($T_n \to \infty$ as $n \to \infty$), solve the approximate null-controllability problem at a free time.

THANK YOU FOR YOUR ATTENTION!