Analogs of Szegö Theorem for Ergodic Operators

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Consider a selfadjoint operator in $l^2(\mathbb{Z})$ (discrete convolution)

$$(Au)_j = \sum_{k\in\mathbb{Z}} A_{j-k}u_k, \ \overline{A_j} = A_{-j}, \ \sum_{j\in\mathbb{Z}} |A_j| < \infty.$$

Let

- $\Lambda = [-M, -M+1, ..., M] \subset \mathbb{Z}$ be an interval,
- $A_\Lambda = \{A_{j,k\in\Lambda}\}$ be the restriction of A to Λ,
- $$a(p) = \sum_{j\in\mathbb{Z}} A_j e^{2\pi i p j} > 0, \ p \in \mathbb{T} = [0, 1)$$

  be the Fourier transform (symbol) of A,
- $\{l_j\}_{j\in\mathbb{Z}}$ be the inverse Fourier transform of $\log a$.

Then (*Szegö 1915 (leading term), 1935 (subleading term)*)

$$\log\det A_\Lambda = |\Lambda| l_0 + \sum_{j=1}^{\infty} j l_j l_{-j} + o(1), \ |\Lambda| \to \infty,$$

where $|\Lambda| = 2M + 1 := L$ and a is smooth enough.
Use the identity $\log\det A_\Lambda = \operatorname{tr}\log A_\Lambda$ to write a "spectral" form

$$\operatorname{tr}\log A_\Lambda = |\Lambda| l_0 + \sum_{j=1}^{\infty} j l_j l_{-j} + o(1), \ |\Lambda| \to \infty,$$

i.e., a two-term asymptotic trace formula for A_Λ via the "limiting" operator A.

This suggests a generalization of the formula, in which log is replaced by a function $\varphi : \mathbb{R} \to \mathbb{C}$ of a certain class.

Generalizations include the multidimensional discrete and continuous cases of $\Lambda \in \mathbb{Z}^d$, $\mathbb{R}^d$, where Λ is, say, a cube of side length L centered in the origin and a and φ are smooth enough

$$\mathrm{tr}\ \varphi(A_\Lambda) = L^d \int_{\mathbb{T}} \varphi(a(p))dp + L^{d-1}T_2 + o(L^{d-1}),\ L \to \infty,$$

where T_2 is an L-independent functional of φ and a.

Observe that the leading term of the Szëgo formula is proportional to the "volume" L^d of Λ while the subleading term is proportional the surface area L^{d-1} of Λ, quite natural from statistical mechanics point of view.

A. Böttcher, B. Silbermann, Analysis of Toeplitz Operators, Springer, 1990
B. Simon, Szegö's Theorem and Its Descendants, PUP, 2011,
SPB: I.A. Ibragimov, A.Laptev, Yu.Safarov, A. Sobolev 60' – 13'
Kharkov: N.I. Akhiezer 60'

It is important to stress that while the leading term of Szëgo formula is fairly insensitive to the smoothness of φ and a, the sub-leading term is not.

An example: $\varphi \in C^\infty$ but a is the indicator of an interval $\Delta \subset \mathbb{T}$. In this case (*Widom 82, Sobolev 12*)

$$\begin{aligned} \mathrm{tr}\ \varphi(A_\Lambda) &= L^d\left((1-|\Delta|)\varphi(0)+|\Delta|\varphi(1)\right) \\ &+ S_2\ L^{(d-1)}\log L + o(L^{(d-1)}\log L),\ L\to\infty. \end{aligned}$$

The case where $\varphi(0)=\varphi(1)$ and $\varphi \in C_\alpha,\ \alpha \in (0,1)$ is important for quantum information theory (violation of the area law in extended translation invariant quantum systems).

A natural generalization of convolution operators in $l^2(\mathbb{Z}^d)$ and $L^2(\mathbb{R}^d)$ are ergodic operators, a well known example is the Schrödinger operator with ergodic potential, see e.g.
L. Pastur, A. Figotin, Spectra of Random and Almost Periodic Operators, Springer, 1992.

Consider the technically simplest case of $l^2(\mathbb{Z})$. Let $(\Omega, \mathcal{F}, P)$ be a probability space, T be a measure preserving and ergodic automorphism of Ω and $A : \Omega \to \mathcal{B}(l^2(\mathbb{Z}))$.
We say that a random operator $A(\omega) := \{A_{jk}(\omega)\}_{j,k\in\mathbb{Z}}$ is ergodic if with probability 1 and for any $t \in \mathbb{Z}$

$$A_{j+t,k+t}(\omega) = A_{jk}(T^t\omega), \ \forall j,k \in \mathbb{Z}.$$

- Convolution operators: take $\Omega = \{0\}$, in particular the operator of second difference (one dimensional discrete Laplacian)

$$(H_0 u)_j = u_{j-1} + u_{j+1}.$$

- The operator V of multiplication $(Vu)_j = v_j u_j,\ j \in \mathbb{Z}$ by ergodic sequence $v = \{v_j\}_{j\in\mathbb{Z}}$, i.e., $\Omega = \mathbb{R}^{\mathbb{Z}},\ (Tv)_j = v_{j+1}$ is the shift and $v_j(\omega) = \mathcal{V}(T^j\omega)$ with a bounded measurable $\mathcal{V} : \Omega \to \mathbb{R}$.
- One dimensional discrete Schrödinger operator

$$H = H_0 + V$$

and now V is called the ergodic potential.

- $\Omega = \mathbb{T}$, $\mathcal{F}$ is the Borel algebra of $\mathbb{T}$, P is the normalized to unity Lebesgue measure on $\mathbb{T}$ and $T\omega \equiv \omega + \alpha \pmod 1$ with an irrational $\alpha \in [0, 1)$. Given $\mathcal{V} : \mathbb{T} \to \mathbb{R}$ (1-periodic), set $v_j = \mathcal{V}(\alpha j + \omega)$ and obtain a simplest almost periodic (quasiperiodic) potential.
- $\Omega = \mathbb{R}^{\mathbb{Z}}$, $\mathcal{F}$ is the σ-algebra of cylinders in $\mathbb{R}^{\mathbb{Z}}$, P is the product measure of a 1d probability law F and $T\{v_j\}_{j\in\mathbb{Z}} = \{v_{j+1}\}_{j\in\mathbb{Z}}$, $v_j = v_0(T^j)$ i.e., $\mathcal{V} = v_0$ and V is the double infinite sequence of i.i.d. random variables. This is a random potential.

An analog could be as follows (again in the 1d case for simplicity). Let B be a selfadjoint ergodic operator in $l^2(\mathbb{Z})$, $a : \mathbb{R} \to \mathbb{C}$ and $\varphi : \mathbb{C} \to \mathbb{C}$ be sufficiently "good" functions. Then $A = a(B)$ is a normal ergodic operator. Denote A_Λ the restriction of A to $l^2(\Lambda), : \Lambda = [-M, M]$. We are again interested in the asymptotic behavior of

$$\mathrm{tr}\, \varphi(A_\Lambda), \ L := 2M + 1 \to \infty,$$

i.e., a linear statistics of the eigenvalues of A_Λ as $|\Lambda| \to \infty$.
The behavior is determined by the triple

$$(B, a, \varphi)$$

of underlying ergodic operator B and functions $a : \mathbb{R} \to \mathbb{C}$, the symbol, and $\varphi : \mathbb{C} \to \mathbb{C}$, the test function.

To make the analogy more clear, consider a convolution operator A and assume for simplicity that its symbol a is even. Then

$$a(p) = \widetilde{a}(\cos 2\pi p), \;\; p \in \mathbb{T}$$

and since $\cos 2\pi p$ is the symbol of the convolution operator H_0 (one-dimensional discrete Laplacian), we can write A as

$$A = \widetilde{a}(H_0).$$

Thus, replace H_0 by the one-dimensional discrete Schrödinger operator $H_0 + V$ with ergodic potential to obtain an interesting class of ergodic operators.

The leading term of $\mathrm{tr}\,\varphi(A_\Lambda)$ for ergodic operators is known.

Recall the notion of the Integrated Density of States (IDS) of an ergodic operator A. Let

$$N_\Lambda^A = |\Lambda|^{-1} \sum_l \delta_{\lambda_l^{(\Lambda)}}$$

be the Normalized Counting Measure of eigenvalues $\{\lambda_l^{(\Lambda)}\}_l$ of A_Λ. Then there exists a non-random non-negative measure N^A known as the Integrated Density of States (IDS) of A and such that for any continuous and bounded function $\varphi : \mathbb{R} \to \mathbb{C}$ with probability 1

Szegö Theorem. Results

Leading Term

Thus

$$\lim_{\Lambda} |\Lambda|^{-1} \mathrm{tr}\, \varphi(A_\Lambda) = \lim_{\Lambda\to\infty} \int \varphi(\lambda) N_\Lambda^A(d\lambda)$$
$$= \int \varphi(\lambda) N^A(d\lambda) = \int \varphi(a(\lambda)) N^B(d\lambda).$$

This implies for $A = a(H)$ with probability 1:

$$\mathrm{tr}\, \varphi(A_\Lambda) = \mathrm{tr}\, \varphi(a_\Lambda(H)) = |\Lambda| \int \varphi(\lambda) N_\Lambda^A(d\lambda)$$
$$= \quad |\Lambda| \int \varphi(\lambda) N^A(d\lambda) + o(|\Lambda|) = \int \varphi(a(\lambda)) N^B(d\lambda) + o(\Lambda)$$
$$= \quad |\Lambda| \mathbf{E}\{\varphi_{00}(A)\} + o(|\Lambda|) = |\Lambda| \mathbf{E}\{(\varphi(a(H)))_{00}\} + o(|\Lambda|), \ |\Lambda| \to \infty.$$

Theorem

Let H be the one dimensional discrete Schroedinger operator with quasiperiodic potential: $V = \{v_j\}_{j\in\mathbb{Z}}$, $v_j = \mathcal{V}(\alpha j + \omega)$, $\mathcal{V} \in C^{[\beta]+2}$ *and* $\alpha \in (0,1)$ *is Diophantine, i.e.,* $|\alpha l - m| \geq C/l^{\beta}$, $\beta > 1$, $\forall m \in \mathbb{Z}, \forall l \in \mathbb{N}$. *Then we have* $\forall z, \mathrm{dist}\{z, \sigma(H)\} \geq \eta_0 > 2$, $\Lambda = [-M, M]$

$$\sum_{|j|\leq M} (G_\Lambda(\omega))_{jj} = |\Lambda| \int_{\mathbb{T}} G_{00}(\omega) d\omega$$
$$+ r_+(\alpha M + \omega) + r_-(-\alpha M + \omega) + o(1), \; M \to \infty,$$

where $r_\pm$ are continuous 1-periodic functions.

Thus, the $O(1)$ subleading terms are as in classical case, however they are almost periodic ("backward" and "forward").

The leading term in the above "stochastic Szegö theorem is of the form of the Law of Large Numbers, i.e., of the order $|\Lambda|$ and non-random. Thus a natural guess is that if eigenvalues of $\varphi(a_\Lambda(H))$ are random enough, then the subleading term is of the form the Central Limit Theorem, i.e., of the order $|\Lambda|^{1/2}$ and Gaussian distributed, but not the $O(1)$ surface term.

Indeed, consider for simpliciry the case where
$a(\lambda) = \lambda, \ \varphi(\lambda) = (\lambda - z)^{-1}$, i.e.,

$$\varphi(a(H)) = G := (H - z)^{-1}, \ \varphi(a_\Lambda(H)) = G_\Lambda := (H_\Lambda - z)^{-1}$$

are the resolvents of H and of its restriction H_Λ.

Theorem

Let $H = H_0 + V$ be the Schrödinger operator whose potential is a sequence of bounded i.i.d. random variables

$$V = \{v_j\}_{j \in \mathbb{Z}}, \ |v_j| \leq V_0.$$

Assume $z = x \in \mathbb{R}$ and $2(V_0 + 1)/|x| \in (0, 1)$. Then the random variable

$$|\Lambda|^{-1/2}(\mathrm{tr}\ G_\Lambda - |\Lambda|\mathbf{E}\{G_{00}\})$$

converges in distribution as $|\Lambda| := L \to \infty$ to the Gaussian random variable γ of zero mean and non-zero finite variance $\sigma^2 > 0$.

Thus, the subleading term is now $|\Lambda|^{1/2}\gamma$ (of the order $|\Lambda|^{1/2}$ and random) but not just independent of $|\Lambda|$ as in the classical Szegö case.

Remarks. i) It is known that

$$\sigma(H_\Lambda) \subset \sigma(H) = [-2 + V_0,\ 2 + V_0)],\ \forall\Lambda.$$

Thus the condition on x guarantees that the theorem is an analog of the smooth case of the Szëgo theorem.
ii) An analogous result is valid for certain classes of $\varphi \circ a \in C^1$.
iii) It is of interest to find the "surface" $(O(1))$ term (now "subsubleading"):

$$s_+(T^M\omega) + s_-(T^{-M}\omega) + O(e^{-2bM}),\ M \to \infty$$

where $\Lambda = [-M, M]$ and the "forward" and "backward" terms $s_\pm$ are

$$s_{\pm}(\omega) = -(1 - G_{0,\pm 1}(\omega))^{-1} \sum_{j=\mp\infty}^{0} G_{0,j}(\omega) G_{j,\pm 1}(\omega).$$

Note that the terms are random (cf. the almost periodic case).

It is worth mentioning that there was no a "serious" use of the spectrum structure (ac, pp) of H so far. This, however, proves to be important the cases where an $O(1)$ term is either leading or subleading, which involve non-smooth symbols.

Consider $a(\lambda) = \chi_\Delta(\lambda)$, $\Delta \in \sigma(H)$ and $\varphi(\lambda) = \lambda(1-\lambda)$, hence $A = P := \mathcal{E}_H(\Delta)$ and

$$\varphi(a(H)) = P(1-P) = 0, \; \varphi(a_\Lambda(H)) = P_\Lambda(1_\Lambda - P_\Lambda).$$

The example is related to the area law of quantum informatics (a toy model)

We will use the following manifestation of the pure point spectrum (Anderson localization) for one dimensional discrete Schrödinger operator with random potential

$$\mathbf{E}\{|P_{jk}|\} \leq Ce^{-\gamma|j-k|}, \; C < \infty, \gamma > 0.$$

The exponential (!?) bound is valid, in particular, if the probability law of i.i.d. random potential has a bounded density.

Theorem

Let H be the Schrödinger operator with random potential such that the above exponential bound holds and $\Delta \in \sigma(H)$, $N^H(\Delta) \in (0,1)$ where N^H is the IDS of H. Then with probability 1

$$\mathrm{tr} P_\Lambda(\mathbf{1}_\Lambda - P_\Lambda) = t_+(T^M\omega) + t_-(T^{-M}\omega) + o(1), \ M \to \infty.$$

$$t_+ = \sum_{j=-\infty}^{0} \sum_{k=1}^{\infty} |P_{jk}|^2, \ t_- = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{-1} |P_{jk}|^2.$$

are non-zero random variables.

Remarks. i) No "volume" contribution, only "surface" one (a "toy" case of the area law of quantum informatics)

ii) $V = 0: \ P_{jk} = \sin c(\Delta)|j-k|/|j-k|, \ \ t_\pm^{(L)} = O(\log L)$, i.e., the Widom-type asymptotics (violation of the area law).

This an important topic of quantum information theory dealing with

$$S_\Lambda = \operatorname{tr} h(P_\Lambda),\ h(x) = -x\log_2 x - (1-x)\log_2(1-x),$$

i.e., with the case of Szegö theorem where $\varphi = h$ (non-smooth!), $a = \chi_\Delta$.

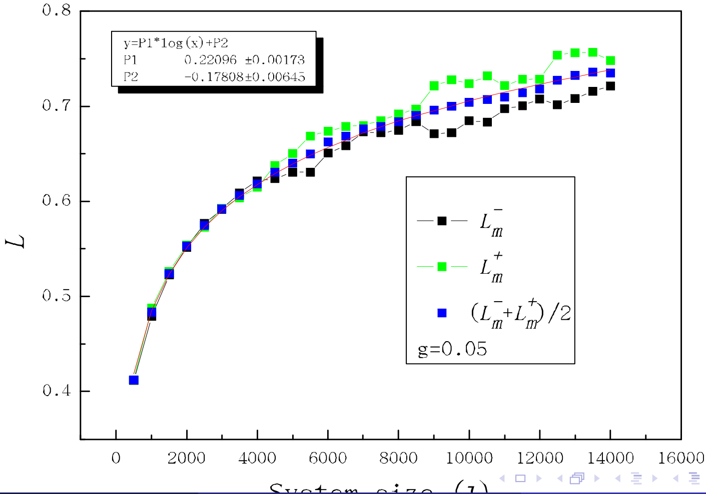
(i) Constant potential, moreover, convolution operators: *Leschke, Sobolev, Spitzer 13*

$$S_\Lambda = C_1 \log L + O(1),\ L = |\Lambda| = 2M + 1 \to \infty,$$

(ii) Random potentials: *P., Slavin 14, Elgart, P. Shcherbina 16.*

$$S_\Lambda = C_2 + o(1),\ L = |\Lambda| = 2M + 1 \to \infty,$$

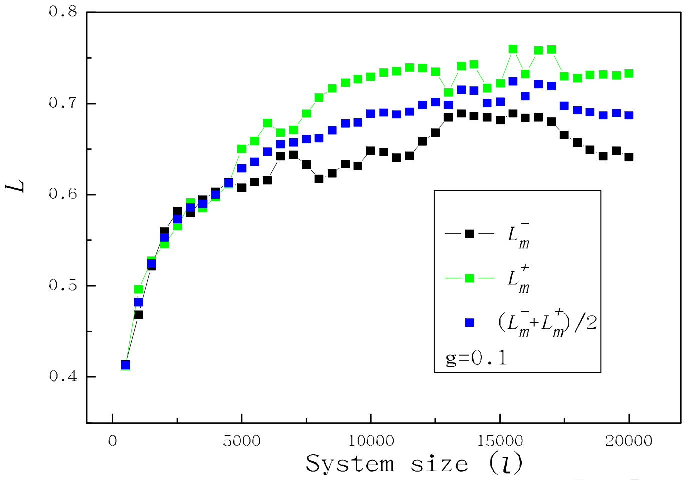
Randomness kills quantum correlations (entanglement).



y=P1*log(x)+P2
P1 0.22096 ±0.00173
P2 -0.17808±0.00645
0.8
0.7
0.6
0.5
0.4
L
L_m^-
L_m^+
$(L_m^- + L_m^+)/2$
g=0.05
0
2000
4000
6000
8000
10000
12000
14000
16000

Quantum Informatics. Emergence of the Area Law

Stronger Disorder



(i) Resolvent identity. Given selfadjoint and invertible A_1 and A_2:

$$(A_1 + A_2)^{-1} = (A_1 + A_2)^{-1} + A_1^{-1}(A_1 - A_2)A_2^{-1}.$$

(ii) If $H = H_0 + V$, $|v_j| \leq V_0$, $\sigma(H) \subset [-2 - V_0, 2 + V_0]$,
$G = (H - z)^{-1}$, $||G|| \leq \delta^{-1}$, and $\delta := \mathrm{dist}(z, [-2 - V_0, 2 + V_0]) > 0$,
then

$$|G_{j,k}| \leq C(z)e^{-b(z)|j-k|}, \; C < \infty, \; b > 0, \; j, k \in \mathbb{Z}.$$

Use (i) with $A_1 = H_0$, $A_2 = V - z$, $||H_0|| = 2$, $||V - z|| \geq \mathrm{dist}(z, [-V_0, V_0]) := \alpha > 2$, $\Leftrightarrow \delta > 0$ to write

$$G_{jk} = \sum_{l=0}^{\infty}((H_0(V - z)^{-1})^l(V - z)^{-1})_{jk} = \sum_{l=|j-k|}^{\infty} \leq (2/\alpha)^{|j-k|}(\alpha - 2)^{-1}.$$

thus $C = 2/(\alpha - 2)^{-1} < \infty$, $b = \log \alpha/2 > 0$.

(iii) If $G_\Lambda = (H|_\Lambda - z)^{-1}$, $\Lambda = [-M, M]$, $\delta > 0$, then

$$(G_\Lambda)_{jk} = G_{jk} - G_{j,M+1}(G_\Lambda)_{Mk} - G_{j,-M-1}(G_\Lambda)_{-M,k}, \; j, k \in \Lambda.$$

Use (i) with $A_1 = H$, $A_2 = H_\Lambda \oplus H_{\mathbb{Z}\setminus\Lambda}$.

(iv) If $\delta > 1$, then

$$\begin{aligned}(G_\Lambda)_{Mk} &= -G_{Mk}(1 + G_{M,M+1})^{-1} + O(e^{-2bM}), \; M \to \infty,\\ (G_\Lambda)_{-Mk} &= -G_{-Mk}(1 + G_{-M-1,-M})^{-1} + O(e^{-2bM}), \; M \to \infty\end{aligned}$$

Use (iii) with $j = M$, (ii) and $|G_{M,M+1}|, |(G_\Lambda)_{Mk}| \le ||G|| < \delta^{-1}$ to estimate

$$|G_{M,-M-1}(G_\Lambda)_{Mk}(1 + G_{M,M+1})_{Mk}| \le C(\delta - 1)^{-1}e^{-2bM}.$$

(v) Basic formulas

$$(G_\Lambda)_{jk} = G_{jk} - \frac{G_{j,M+1}G_{Mk}}{1+G_{M,M+1}} - \frac{G_{j,-M-1}G_{-M,k}}{1+G_{-M,-M-1}} + O(e^{-2bM}),\ j,k \in \Lambda \to \infty,$$

use (iii) and (iv);

$$\mathrm{tr}\ G_\Lambda = \sum_{j\in\Lambda} G_{jj} + s_+^{(M)} + s_-^{(M)} + O(e^{-b|\Lambda|}),$$

use the previous formula for $j = k$ and (ii) to obtain

$$s_\pm^{(M)} = -(1 + G_{\pm M,\pm(M+1)})^{-1} \sum_{j=-M}^{M} G_{j,\pm(M+1)} G_{\pm Mj}.$$

An example of the convolution operator, classical case of Szegö's theorem. Here $G_{jk} = G_{j-k}$, $G_j = G_{-j}$ ($\{G_{j,k}\}$ is symmetric since H is real), hence, by basic formula (ii)

$$\mathrm{tr}\; G_\Lambda = |\Lambda| G_0 + s_+ + s_- + O(e^{-b|\Lambda|}), \; |\Lambda| = 2M + 1 \to \infty,$$

$$s_\pm = -(1 + G_{\pm 1})^{-1} \sum_{j=\pm\infty}^{0} G_j G_{-j}.$$

This is a simple particular case of Szegö's theorem. To check use

$$G_j = \int_0^1 \frac{e^{2\pi i p j}}{2\cos 2\pi p - z} dp.$$

Since H is ergodic, $G = (H - z)^{-1}$ is also egodic, hence

$$G_{jk}(T^a\omega) = G_{j+a,k+a}(\omega),$$

and by basic formula we obtain the relation

$$\mathrm{tr}\ G_\Lambda = \sum_{j\in\Lambda} G_{jj} + s_+(T^{+M}\omega) + s_-(T^{-M}\omega) + O(e^{-2bM}),$$

$$|\Lambda| = 2M + 1 \to \infty,$$

having again the backward and forward terms à la Szegö and valid with probability 1, where

$$s_\pm(\omega) = -\frac{1}{1 + G_{0,\pm1}(\omega)} \sum_{j=\pm\infty}^{0} G_{j0}(\omega) G_{\pm1,j}(\omega).$$

are well defined random variables.

Indeed, we have by (ii) and by ergodicity

$$\begin{aligned}\sum_{j=-M}^{M} G_{jM}(\omega)G_{M+1,j}(\omega) &= \sum_{j=-\infty}^{M} G_{jM}(\omega)G_{M+1,j}(\omega) + O(e^{-2bM}) \\ &= \sum_{j=-\infty}^{0} G_{j0}(T^M\omega)G_{1,j}(T^M\omega) + O(e^{-2bM})\end{aligned}$$

Besides, by ergodic theorem we have for the first term with probability 1

$$\sum_{j\in\Lambda} G_{jj}(\omega) = \sum_{j\in\Lambda} G_{00}(T^j) = |\Lambda|\mathbf{E}\{G_{00}\} + o(|\Lambda|),\ |\Lambda| \to \infty,$$

thus it gives the leading term à la Szegö, but not more in general!

Here $v_j = \mathcal{V}(\alpha j + \omega)$, $\mathcal{V} \in C^{[\beta]+2}$ is 1-periodic, $\alpha \in (0,1)$ is Diophantine $|\alpha l - m| \geq C/l^{\beta}$, $\beta > 1, \forall m \in \mathbb{Z}, \forall l \in \mathbb{N}$, and $\omega \in [0,1)$ (the "randomness") parameter, hence

$$G_{jj}(\omega) = \mathcal{G}(\alpha j + \omega),\ \mathcal{G}(\omega) := G_{00}(\omega),$$

and (recall H. Weyl)

$$\sum_{j \in \Lambda} G_{jj}(\omega) = \sum_{j \in -M}^{M} \mathcal{G}(\alpha j + \omega).$$

Since $\mathcal{G}$ is 1-periodic and of $C^{[\beta]+2}$, we have by (i)

$$\mathcal{G}(\omega) = \sum_{l \in \mathbb{Z}} \mathcal{G}_l e^{2\pi i \omega l},\ |\mathcal{G}_l| = O(1/|l|^{[\beta]+2}),$$

and

$$\sum_{j \in \Lambda} G_{jj}(\omega) = \sum_{j \in -M}^{M} \mathcal{G}(\alpha j + \omega) = |\Lambda| \mathcal{G}_0 + g_+(\alpha M + \omega) + g_-(-\alpha M + \omega),$$

where

$$g_{\pm}(\omega) = \sum_{l \neq 0} \mathcal{G}_l e^{2\pi i l \omega \pm \pi i \alpha l} / 2i \sin \pi \alpha l$$

and since $|\sin \pi \alpha l| = |\sin \pi(\alpha l - m)| \geq C|l|^{-\beta}$ and $|\mathcal{G}_l| \leq C/l|^{2+[\beta]}$, the series is absolutely convergent.
We obtain finally uniformly in $\omega \in [0, 1)$ and for $|\Lambda| \to \infty$

$$\text{tr } G_{\Lambda} = |\Lambda| \int_0^1 G_{00}(\omega) d\omega + r_+(T^{+M}\omega) + r_-(T^{-M}\omega) + O(e^{-b|\Lambda|}),$$
$$r_{\pm}(\omega) = s_{\pm}(\omega) + g_{\pm}(\omega)$$

The leading term is "nonrandom", since $\int_0^1 G_{00}(\omega) d\omega = \mathbf{E}\{G_{00}\}$ and the subleading terms (à la Szegö and new) are bounded and "almost periodic" in M.

A General CLT (à la S. Bernstein)

Theorem

Let $\{X_j\}_{j\in\mathbb{Z}}$ be i.i.d. random variables and Y_0 be a bounded Borelian function of $\{X_j\}_{j\in\mathbb{Z}}$. Assume that $\mathbf{E}\{Y_j\}=0$ and

$$\sum_{p=1}^{\infty}\mathbf{E}\{|Y_p-\mathbf{E}\{Y_p|\mathcal{F}_{-p}^{p}\}|\}<\infty.$$

where $\mathcal{F}_a^b$ is the σ-algebra generatd by $\{X_j\}_{j=a}^{j=b}$, $[a,b]\subset\mathbb{Z}$. Then $\sigma^2:=\sum_{j\in\mathbb{Z}}\mathbf{E}\{Y_0Y_j\}<\infty$ and if $\sigma^2>0$ the normalized sum $(2M+1)^{-1/2}\sum_{j=-M}^{M}Y_j$ converges in distribution to the Gaussian random variable γ such that $\mathbf{E}\{\gamma\}=0$ and $\mathbf{Var}\{\gamma\}:=\mathbf{E}\{\gamma^2\}-\mathbf{E}^2\{\gamma\}=\sigma^2$.

For the above theorem see *I.A.Ibragimov, Yu.V.Linnik Independent and Stationary Sequences of Random Variables, Wolters-Noordhoff, Groningen, 1986.*

The theorem conditions are:
(a) the decay of correlations

$$\sum_{p=1}^{\infty} \mathbf{E}\{|Y_p - \mathbf{E}\{Y_p|\mathcal{F}_{-p}^{p}\}|\} < \infty;$$

(b) the positivity of the variance σ^2.

We take $X_j = v_j, \; Y_0 = G_{00}^{\circ} = G_{00} - \mathbf{E}\{G_{00}\}$.

To check the condition of decay of correlations, set $G^{(p)} = G|_{v_j=0,\ |j|>p}$ and use the resolvent identity for $R_p = G_{00} - G_{00}^{(p)}$ and (ii):

$$|R_p| = \left| \sum_{|j|>p} G_{0j} v_j G_{j0}^{(p)} \right| \leq V_0 \delta^{-1} \sum_{|j|>p} |G_{0j}| = O(e^{-bp}).$$

Since $\mathbf{E}\{G_{00}^{(p)}|\mathcal{F}_{-p}^{p}\} = G_{00}^{(p)}$, we have

$$\mathbf{E}\{|Y_p - \mathbf{E}\{Y_p|\mathcal{F}_{-p}^{p}\}|\} = \mathbf{E}\{|R_p - \mathbf{E}\{R_p|\mathcal{F}_{-p}^{p}\}|\} = O(e^{-bp}).$$

Theorem

Let $\{\xi_j^t\}_{j=1}^N$, $t \in I$ *be i.i.d. random variables whose common probability law has a density* f_t, $\varphi : \mathbb{R}^N \to \mathbb{R}$ *and* $\Phi_t = \varphi(\xi_1^t, \dots, \xi_N^t)$. *Then*

$$\mathbf{Var}\{\Phi_t\} \ : \ = \mathbf{E}\{\Phi_t^2\} - \mathbf{E}^2\{\Phi_t\} \geq \left(\frac{d}{dt}\mathbf{E}\{\Phi_t\}\right)^2 \Big/ NF_t$$

where

$$F_t = \int \left(\frac{d}{dt}\log f_t(x)\right)^2 f_t(x)dx = \int dx \left(\frac{d}{dt} f_t(x)\right)^2 / f_t(x)dx$$

is the Fisher information.

Proof (single variable, $N = 1$). Use the Cauchy-Schwarz inequality

$$\mathbf{Var}\{\eta_1\} \geq (\mathbf{Cov}\{\eta_1\eta_2\})^2/\mathbf{Var}\{\eta_2\}$$

where

$$\begin{aligned}\mathbf{Var}\{\eta\} &= \mathbf{E}\{(\eta - \mathbf{E}\{\eta\})^2\} \\ \mathbf{Cov}\{\eta_1\eta_2\} &= \mathbf{E}\{(\eta_1 - \mathbf{E}\{\eta_1\})(\eta_2 - \mathbf{E}\{\eta_2\})\}.\end{aligned}$$

Take $\eta_1 = \varphi(\xi_t)$, $\eta_2 = \frac{d}{dt}(\log f_t(\xi_t))$. We obtain:

$$\mathbf{E}\{\eta_2\} = \int \frac{f_t^{\cdot}(x)}{f_t(x)} f_t(x)dx = 0, \ \mathbf{Cov}\{\eta_1\eta_2\} = \frac{d}{dt}\int \varphi(x) f_t(x)dx.$$

Szegö Theorem. Proof of CLT

Positivity of Variance

Take $\xi_t = tv_j,\ t \in [1-\varepsilon, 1+\varepsilon]$. Since it is easy to proof that

$$\sigma^2 = \lim_{M\to\infty} \mathbf{Var}\{\Xi_M\},\ \Xi_M = (2M+1)^{-1/2} \sum_{|j|\le M} G_{jj}$$

take $\Phi = \Xi_M$. Then by (i)

$$\frac{d}{dt}\mathbf{E}\{\Xi_M\}|_{t=1} = -(2M+1)^{1/2}\mathbf{E}\{G_{00}^2 v_0\}$$

and

$$F|_{t=1} = \int \frac{(f(x)+xf'(x))^2}{f(x)}dx$$

thus

$$\sigma^2 \ge (\mathbf{E}\{G_{00}^2 v_0\})^2 / F|_{t=1}$$

One needs to prove:

$$\mathbf{E}\{G_{00}^2 v_0\} > 0, \; F_1 > 0.$$

Examples.
(i) $v_0 \geq 0$, since by spectral theorem

$$G_{00}^2 = \int_{\sigma(H)} \frac{(\mathcal{E}_H)_{00}(d\lambda)}{(\lambda - x)^2} > 0, \; x \notin \sigma(H).$$

(ii) $F_1 = 0$. Assume that the support of f is $[a, b]$ and $0 < f < \infty$, $x \in [a, b]$. Then

$$F_1 = 0 \Rightarrow f(x) + xf(x) = 0 \Rightarrow f(x) = -C \log x, \; [a, b] \subset [0, 1].$$