The Anderson Model on the Bethe Lattice: Lifshitz Tails

Christoph Schumacher

TU Chemnitz, Germany

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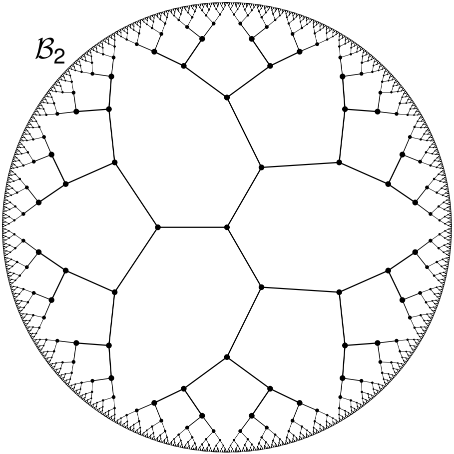
joint work with Francisco Hoecker-Escuti

The Bethe lattice
with coordination number 3

- The Bethe lattice $\mathcal{B}_k$ is an infinite simple tree graph of constant degree $k+1 \geq 3$
- Cayley graph of non-abelian group $\langle a_0, \ldots, a_k \mid a_j^2 = 1 \rangle$
- introduced 1935 by Hans Bethe



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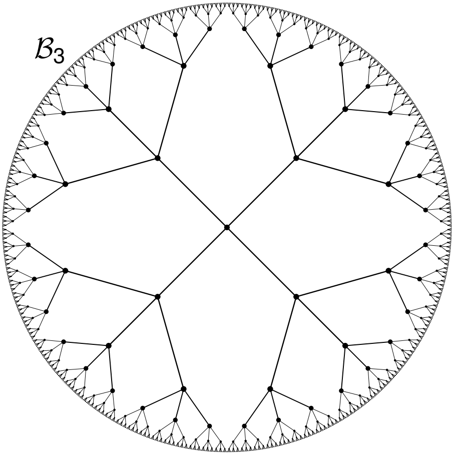

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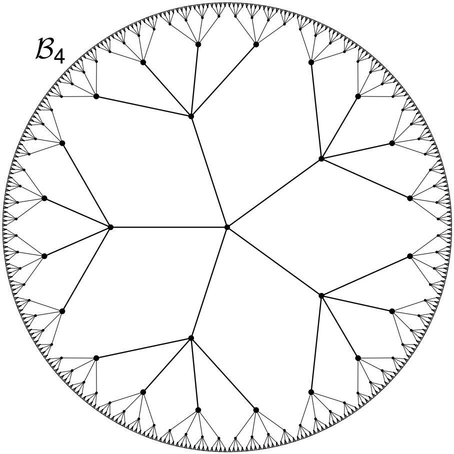

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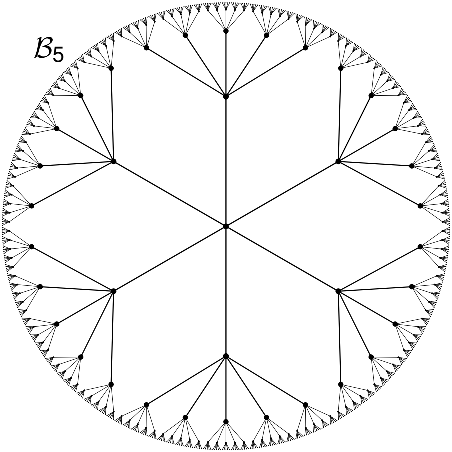

The discrete Anderson model

- Γ: infinite simple undirected graph,e. g. $\Gamma = \mathbb{Z}^d$ or $\Gamma = \mathcal{B}_k$
- The discrete Laplace operator on (the nodes of) Γ:

$$\Delta_\Gamma : \ell^2(\Gamma) \to \ell^2(\Gamma), \qquad (\Delta_\Gamma \varphi)(v) := \sum_{w \sim v} \big(\varphi(w) - \varphi(v)\big)$$

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- Ergodicity implies almost sure spectrum $\Sigma = \sigma(H_\omega^\Gamma)$ a. s.

The Integrated Density of States

The Integrated Density of States (IDS)

$$\mathcal{N}^\Gamma : \mathbb{R} \to [0,1], \quad \mathcal{N}^\Gamma(E) := \mathbb{E}[\langle \delta_v, \mathbf{1}_{(-\infty,E]}(H_\omega^\Gamma)\delta_v \rangle]$$

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- contains spectral information,e. g. $\mathcal{N}^\Gamma$ is a distribution function, and support of the corresponding measure$= \Sigma = \sigma(H_\omega^\Gamma)$ a. s.,
- encodes geometric properties of the underlying space, e. g. $E_0 := \inf \Sigma = 0 \iff \Gamma$ is amenable.

For the Bethe lattice: $E_0 := \inf \Sigma = (\sqrt{k} - 1)^2 > 0$.

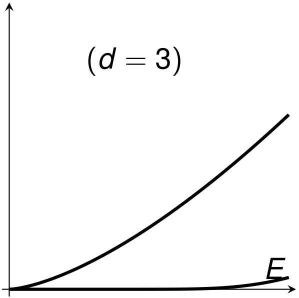
Behavior of the IDS

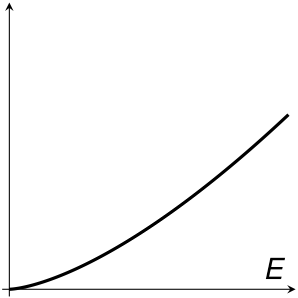
at the bottom of the spectrum

Anderson Hamiltonian: $H^{\Gamma}_{\omega} := -\Delta_{\Gamma} + \lambda V^{\Gamma}_{\omega}$

$\mathcal{N}^{\Gamma}(E_0 + E)$ "$\sim$"	Euclidian lattice $\Gamma = \mathbb{Z}^d$	Bethe lattice $\Gamma = \mathcal{B}_k$
$\lambda = 0$	$E^{d/2}$	$E^{3/2}$
$\lambda > 0$	$\exp(-E^{-d/2})$	

$\mathcal{N}^{\Gamma}(E_0 + E)$





Behavior of the IDS and the DoS

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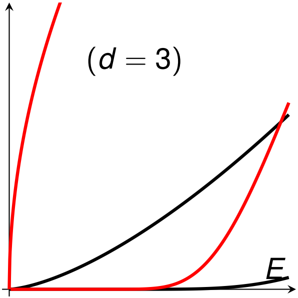

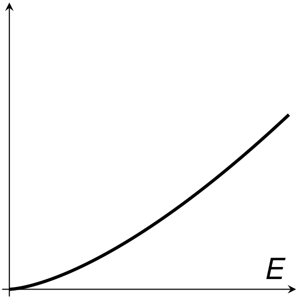

Lifshitz tails

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Theorem (Lifshitz tails on the Bethe lattice)

Assume $\nu := \limsup_{\kappa \searrow 0} \kappa^{1/2} \log \left| \log \mathbb{P}(\omega_v \le \kappa) \right| < 1 \qquad (v \in \mathcal{B}_k)$*.*
Then there exists $\varepsilon > 0$ *such that, for all* $E \in (0, \varepsilon)$*,*

$$\exp\left(-\exp(\varepsilon^{-1} E^{-1/2})\right) \le \mathcal{N}^{\mathcal{B}_k}(E_0 + E) \le \exp\left(-\exp(\varepsilon E^{-1/2})\right)$$

and thus

$$\lim_{E \searrow 0} \frac{\log \log |\log \mathcal{N}^{\mathcal{B}_k}(E_0 + E)|}{\log(E)} = -\frac{1}{2}.$$

Note: $\nu < 1$ is an assumption on the distribution of the potential: roughly: $\mathbb{P}(\omega_v \le \kappa) \gg \exp(-\exp(\kappa^{-1/2}))$ as $\kappa \searrow 0$,
i. e. small values are not too improbable.

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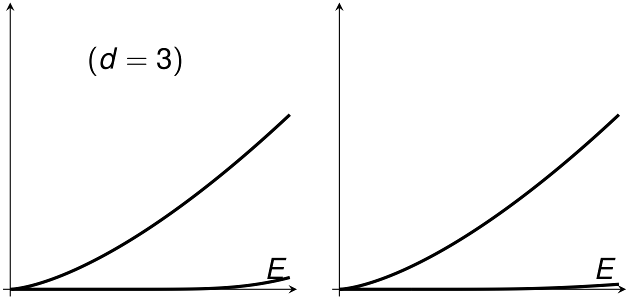

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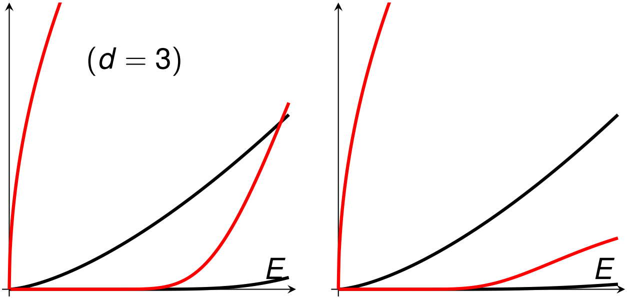

Comparison Euclidian lattice and Bethe lattice

Tools on $\mathbb{Z}^d$:

- amenability – approximation by finite balls: $\mathbb{Z}^d$: $\checkmark$, $\mathcal{B}_k$: $\mathcal{X}$
- perturbation theory – large spectral gap: $\mathbb{Z}^d$: $\checkmark$, $\mathcal{B}_k$: $\mathcal{X}$
- Fourier transform – abelian group: $\mathbb{Z}^d$: $\checkmark$, $\mathcal{B}_k$: $\mathcal{X}$

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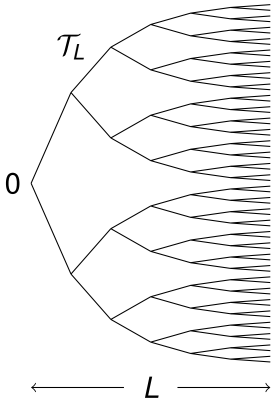
We use

- Laplace transform of IDS $\mathcal{N}^{\mathcal{B}_k}$, Tauberian theorem
- discrete Feynman–Kac formula
- discrete Ismagilov–Morgan–Sigal formula

to reduce Lifshits tails behaviour to properties of ground state energies of Anderson models on finite symmetric rooted trees.

Finite symmetric rooted trees $\mathcal{T}_L$

- Number of children: $k \geq 2$
- Length of tree: $L \in \mathbb{N}$
- root: 0
- Level of node $v \in \mathcal{T}_L$:
  $|v| = \text{dist}(0, v) + 1$
- Advantage w. r. t. $\mathcal{B}_k$:
  explicit formulas for all eigenfunctions and -values



Estimation of the random ground state energy

Consider

- Anderson model $H_\omega^{\mathcal{T}_L} := -\Delta_{\mathcal{T}_L} + V_\omega^{\mathcal{T}_L}$ on the tree $\mathcal{T}_L$
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Theorem (random ground state energy on trees)

Assume $\nu < 1$. Then there are $\varepsilon > 0$ and $L^* > 1$ such that for all $L > L^*$ we have

$$\mathbb{P}\Big(E_0 + \frac{\varepsilon}{(\log L)^2} \leq E_{GS}^L \leq E_0 + \frac{\varepsilon^{-1}}{(\log L)^2}\Big) \geq 1 - \exp(-\varepsilon L).$$

(As before: $E_0 := (\sqrt{k} - 1)^2$)

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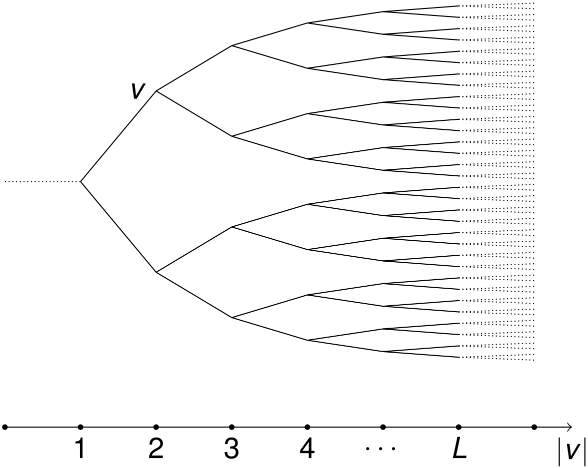
Adjacency operator:

$$A\colon \ell^2(\mathcal{B}_k) \to \ell^2(\mathcal{B}_k), \qquad (A\varphi)(v) := \sum_{w \sim v} \varphi(w), \qquad A = \Delta_{\mathcal{B}_k} + k + 1$$

The ground state on finite symmetric rooted trees

or: radially symmetric eigenfunctions of the adjacency matrix



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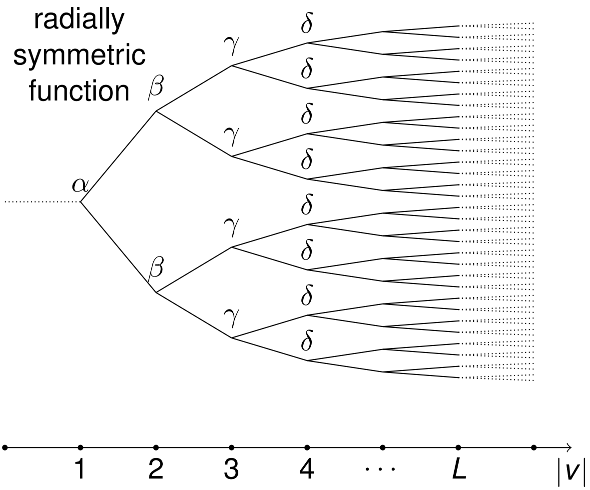

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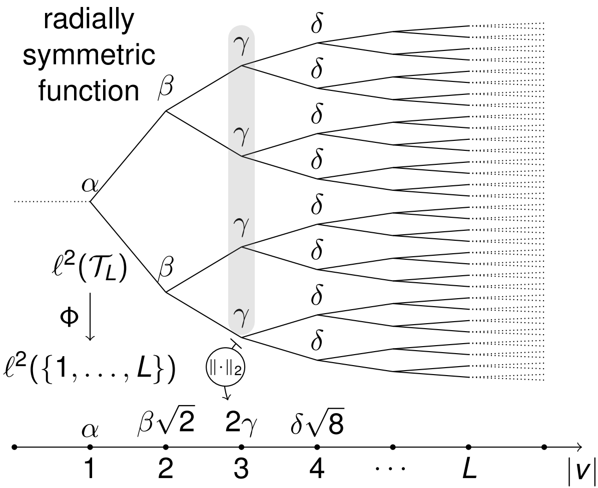

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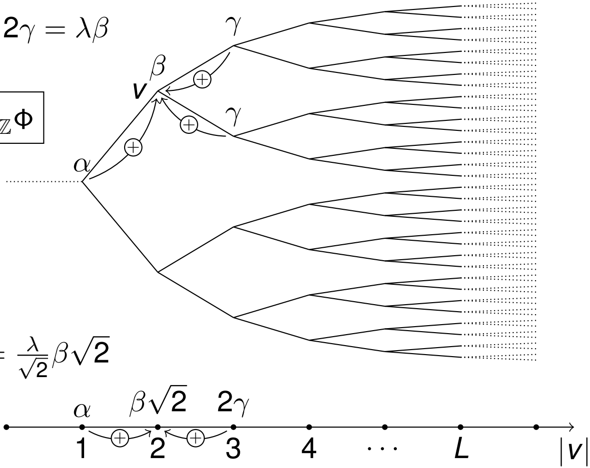
$$\alpha + 2\gamma = \lambda\beta$$

$$\Phi A = \sqrt{k} A_{\mathbb{Z}} \Phi$$



$$\alpha + 2\gamma = \frac{\lambda}{\sqrt{2}} \beta \sqrt{2}$$

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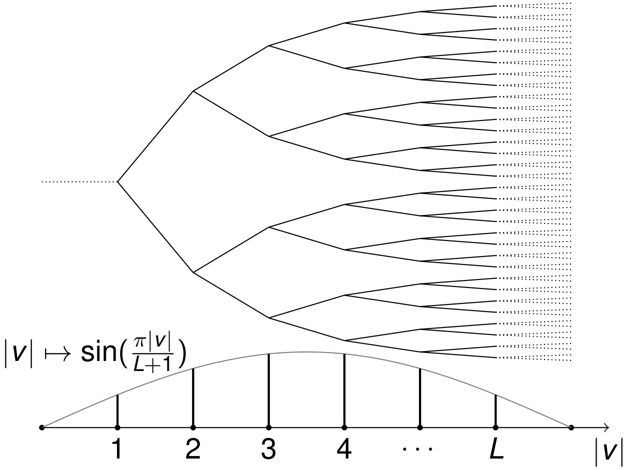

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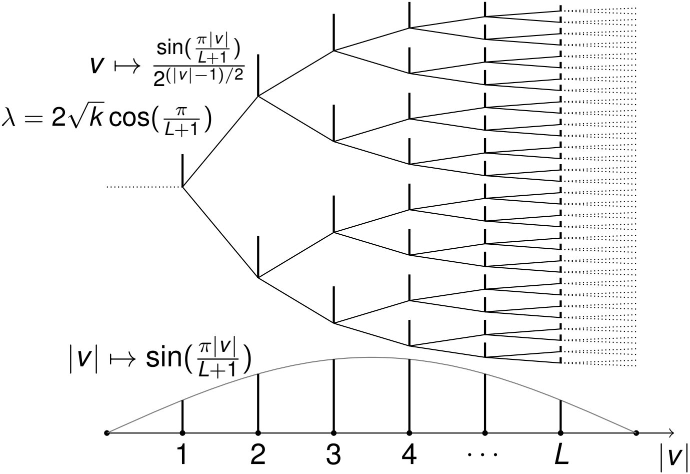

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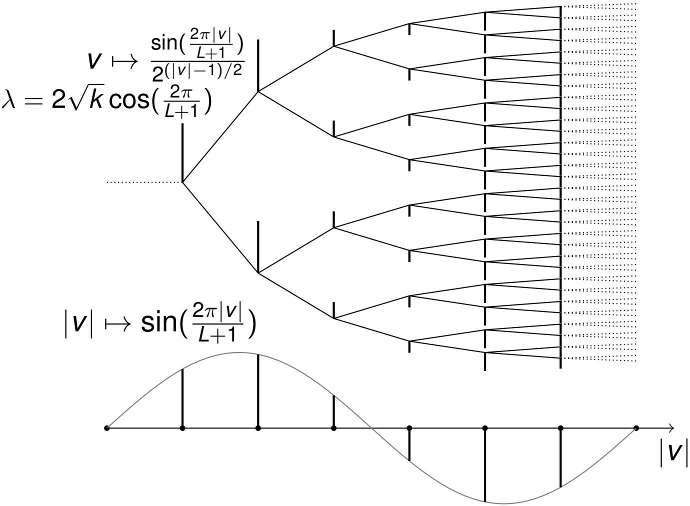
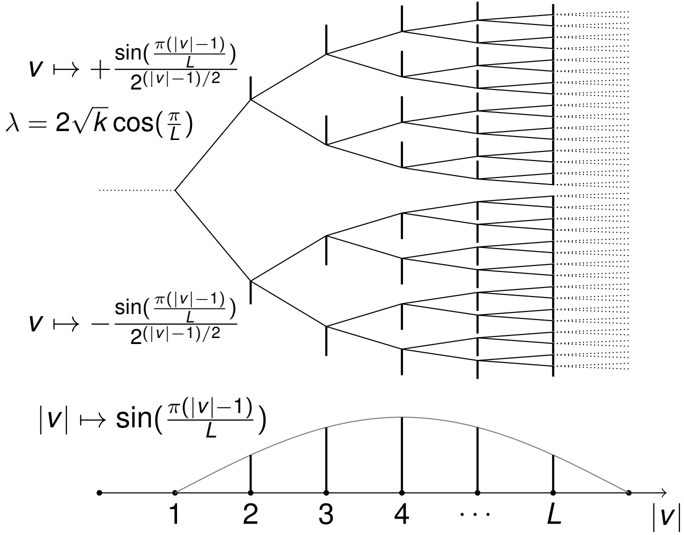


Non-radially symmetric eigenfunctions



Consequences of the spectral properties of trees

- spectral gap:$|\cos(\frac{\pi}{L+1}) - \cos(\frac{\pi}{L})| \approx L^{-3}$on $\mathcal{B}_k$
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- low energy eigenfunctions suppress boundary effects
- location of the random ground state

Thank you for your attention!