Linear Factors and Stabilizers of Binary Forms

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Summary Each binary form F decomposes into linear factors. This decomposition has consequences for the stabilizer of F in the transformation group GL_2. For instance the stabilizer is finite if F has at least three essentially different linear factors.

1 Factorization of Binary Forms

Let k be an algebraically closed field and T be an indeterminate. Every non-constant polynomial $f \in k[T]$ decomposes into linear factors:

$$f = \prod_{i=1}^{n} l_i$$

where the $l_i = a_i + b_i T \in k[T]$ are polynomials of degree 1, in particular $b_i \in k^\times$, and $n = \deg f$. This decomposition is unique up to the order of the factors and up to scalar multipliers $\in k^\times$.

> In other words, the polynomial ring $k[T]$ is factorial (or UFD), the linear polynomials are its prime elements, and the non-zero constants are its units. (By abuse of terminology we use the term "linear" for polynomials as synonymous with "of degree 1".)

The linear factors l_i are not necessarily different. A linear factor $l \mid f$ has multiplicity r if $l^r \mid f$ and $l^{r+1} \nmid f$.

Now we consider binary forms over k, that is, homogeneous polynomials $F \in k[X,Y]$ in two indeterminates X and Y. They have the form (where n is the degree)

$$F = \sum_{\nu=0}^{n} a_\nu X^{n-\nu} Y^\nu = X^n \cdot \sum_{\nu=0}^{n} a_\nu \left(\frac{Y}{X}\right)^\nu = X^n \cdot f(\frac{Y}{X})$$

where $f = \sum a_\nu T^\nu \in k[T]$ is a polynomial of degree $\leq n$. Let $f = \prod_{i=1}^n l_i$ be the decomposition into linear factors—if $\deg f = m < n$, set $l_{m+1} = \cdots = l_n = 1$ constant. From this we get a corresponding decomposition

$$F = \prod_{i=1}^n L_i \quad \text{where the } L_i = X \cdot l_i(\frac{X}{Y}) = a_i X + b_i Y$$

are homogenous of degree 1, or binary linear forms. In the case $\deg f = m < n$ we have $L_{m+1} = L_n = X$, hence X is a linear factor of F of multiplicity $n - m$. (Here the term "linear" is used in a correct way.)

2 The Action of the Group GL_2 on Binary Forms

Now we consider the group $G = GL_2(k)$ of 2×2-matrices with non-zero determinant over k. The matrix

$$(1) \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

acts on the 2-dimensional vector space k^2 by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Denote the coordinate functions $k^2 \longrightarrow k$ by X and Y, where

$$X \begin{pmatrix} x \\ y \end{pmatrix} = x, \quad Y \begin{pmatrix} x \\ y \end{pmatrix} = y$$

for all $x, y \in k$. The inverse of g is

$$g^{-1} = \frac{1}{\delta} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $\delta = \det g = ad - bc$. Thus the induced ("contragredient") action on the space of linear forms spanned by the coordinate functions X and Y is given by

$$X \mapsto \frac{d}{\delta} X - \frac{b}{\delta} Y,$$
$$Y \mapsto -\frac{c}{\delta} X + \frac{a}{\delta} Y.$$

(In general a function $f\colon k^2 \longrightarrow k$ is transformed to $f \circ g^{-1}$.)

Let $R = k[X, Y]$ be the polynomial ring and R_n be its homogeneous part of degree n with $\dim_k R_n = n + 1$. The action of GL_2 extends to automorphisms of R that preserve the degree. Thus R_n is a GL_2-invariant subspace of R.

Some elements and subgroups of GL_2

The group GL_2 contains the matrices

$$D(s,t) = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \quad \text{with } s,\, t \in k^\times,$$

$$A(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{with } b \in k,$$

$$I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{that has } I^2 = \mathbf{1},$$

and the subgroups:

$$\begin{aligned}
SL_2 &= \{g \in GL_2 \mid \det g = 1\},\\
T &= \left\{ D(s,t) \mid s,\, t \in k^\times \right\}, \text{ the canonical maximal torus of } GL_2,\\
N &= T \cup IT, \text{ the normalizer of } T \text{ in } GL_2,\\
Z &= \left\{ D(t,t) \mid t \in k^\times \right\}, \text{ the center of } GL_2,\\
Z_n &= \left\{ D(t,t) \mid t^n = 1 \right\}, \text{ a finite cyclic group of order } \mid n,\\
&\qquad \text{(the order equals } n \text{ if and only if } \operatorname{char} k \nmid n, \text{)}\\
Z' &= Z \cap SL_2 = Z_2 = \begin{cases} \{\pm\mathbf{1}\} & \text{if } \operatorname{char} k \neq 2, \\ \{\mathbf{1}\} & \text{if } \operatorname{char} k = 2, \end{cases} \quad \text{the center of } SL_2,\\
U &= \{A(b) \mid b \in k\}, \text{ the canonical maximal unipotent subgroup of } GL_2,\\
B &= \text{the group of invertible upper triangular matrices,}\\
&\qquad \text{the canonical Borel subgroup of } GL_2,\\
B^- &= \text{the group of invertible lower triangular matrices.}
\end{aligned}$$

Furthermore we sometimes consider the group

$$PGL_2 = GL_2/Z \cong SL_2/Z'.$$

3 Representatives of Orbits

In R_1, the space of linear forms, the group GL_2 has exactly two orbits, $\{0\}$ and $R_1^\bullet = R_1 - \{0\}$. In other words,

Proposition 1 *GL_2 (even SL_2) acts transitively on $R_1^\bullet$.*

Proof. Let $L = \alpha X + \beta Y$ be a non-zero linear form, and define $g \in SL_2$ by

$$g^{-1} = \begin{pmatrix} 1/\beta & 0 \\ \alpha & \beta \end{pmatrix}.$$

Then $g \cdot Y = \alpha X + \beta Y = L$ by the formula for the effect on Y. Hence L is in the SL_2-orbit of Y. $\diamond$

An analogous reasoning for the action on the Cartesian product $R_1 \times R_1$ yields a weaker result:

Proposition 2 *Let L_1 and $L_2 \in R_1$ be non-proportional. Then there is a matrix $g \in GL_2$ with $g \cdot Y = L_1$ and $g \cdot X = L_2$.*

Proof. Let $L_1 = \alpha_1 X + \beta_1 Y$ and $L_2 = \alpha_2 X + \beta_2 Y$. The non-proportionality (or linear independence) is equivalent with the determinant condition $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. If we define g by

$$g^{-1} = \begin{pmatrix} \alpha_2 & \beta_2 \\ \alpha_1 & \beta_1 \end{pmatrix} \in GL_2$$

the formulas for the effects on X and Y yield $g \cdot Y = L_1$ and $g \cdot X = L_2$. $\diamond$

Thus $R_1 \times R_1$ consists of the following GL_2-orbits:

- $\{0\}$
- $R_1^\bullet \times \{0\}$, the orbit of $(X, 0)$
- $\{0\} \times R_1^\bullet$, the orbit of $(0, Y)$
- the (infinitely many) "diagonals" $D_c := \{(L, cL) | L \in R_1^\bullet\}$ for arbitrary $c \in k^\times$, the orbits of (Y, cY)
- $R_1^\bullet \times R_1^\bullet - \cup_{c \in k^\times} D_c$, the orbit of (X, Y)

Finally we look for triples of linear forms L_1, L_2, L_3, which we assume as pairwise non-proportional. Having transformed L_1 to Y and L_2 to X we note that only the unit matrix $\mathbf{1}$ fixes both X and Y, so each non-proportional $L \in R_1$ yields a different orbit, represented by (X, Y, L). However if we consider lines kL through the origin $0 \in R_1$, or points $[L]$ of the projective space $\mathbb{P}^1$, we see that the diagonal matrices $D(s, t)$ fix the lines kX and kY, i. e. the corresponding points of $\mathbb{P}^1$. Thus we have more degrees of freedom to transform the third line kL_3:

Proposition 3 *Let L_1, L_2, $L_3 \in R_1$ be pairwise non-proportional. Then there is a matrix $g \in GL_2$ with $g \cdot L_1 \in kY$, $g \cdot L_2 \in kX$, $g \cdot L_3 \in k(X + Y)$.*

Proof. By Proposition 2 we may choose $h \in GL_2$ with $h \cdot L_1 = Y$ and $h \cdot L_2 = X$. Let $h \cdot L_3 = \alpha_3 X + \beta_3 Y$. The diagonal matrix $D(\beta_3^{-1}, \alpha_3^{-1})$ transforms X to $\alpha_3 X$, Y to $\beta_3 Y$, and $X + Y$ to $h \cdot L_3$. Hence $g := D(\beta_3, \alpha_3)\, h$ transforms L_1 to $\beta_3^{-1} Y$, L_2 to $\alpha_3^{-1} X$, and L_3 to $X + Y$. $\diamond$

The pairwise non-proportionality of linear forms means that the corresponding points of the projective space $\mathbb{P}^1$ are different. Thus another way to express Proposition 3 is that the action of GL_2 on $\mathbb{P}^1$ is 3-transitive. The subset

$$W = \{(x, y, z) \mid x \neq y,\ x \neq z,\ y \neq z\} \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

is a Zariski open dense GL_2-stable subset on which GL_2 acts transitive, and W is the orbit of $([X], [Y], [X+Y])$.

Corollary 1 *Assume that $F \in R_n$ has a linear factor of multiplicity n. Then there is a matrix $g \in SL_2$ with $g \cdot F = Y^n$.*

Proof. We have $F = c\,L^n$ with $c \in k^\times$ and $L \in R_1^\bullet$. Since k is algebraically closed c has an n-th root which we may multiply with L and therefore assume that $F = L^n$. Then we choose g with $g \cdot L = Y$ by Proposition 1, hence $g \cdot F = Y^n$. $\diamond$

Corollary 2 *Assume that $F \in R_n$ has two non-proportional linear factors, one of multiplicity r, and another one of multiplicity $n-r$. Then there is a matrix $g \in GL_2$ with $g \cdot F = X^{n-r}Y^r$.*

Proof. We have $F = c\,L_1^r L_2^{n-r}$ with $c \in k^\times$ and non-proportional linear forms $L_1, L_2 \in R_1$. Again we may absorb c into L_1, hence assume that $c = 1$. Then by Proposition 2 we choose g with $g \cdot L_1 = Y$ and $g \cdot L_2 = X$, hence $g \cdot F = Y^r X^{n-r}$. $\diamond$

Corollary 3 *Assume that $F \in R_n$ has at least three pairwise non-proportional linear factors, say of multiplicities q, r, s, thus $F = L_1^q\, L_2^r\, L_3^s\, \tilde{F}$ with $\tilde{F} \in R_{n-q-r-s}$. Then there is a matrix $g \in GL_2$ and a homogeneous polynomial $H \in R_{n-q-r-s}$ with $g \cdot F = X^q\, Y^r\, (X+Y)^s\, H$.*

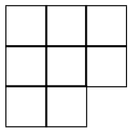
Proof. By Proposition 3 we may choose g with $g \cdot L_1 = c_1 X$, $g \cdot L_2 = c_2 Y$, $g \cdot L_3 = c_3(X+Y)$. Then $g \cdot F = X^q\, Y^r\, (X+Y)^s\, H$ with $H = g \cdot \tilde{F}/c_1 c_2 c_3$. $\diamond$

In the general case we rewrite the factorization of $F \in R_n$ as

$$F = L_1^{l_1} \cdots L_r^{l_r} \quad \text{with } l_1 \geq \ldots \geq l_r > 0$$

where the L_i are pairwise non-proportional linear forms, and $l_1 + \cdots + l_r = n$. Then the action of GL_2 preserves the pattern $(l_1, \ldots, l_r)$.

> This pattern might be interpreted as the shape of a Young diagram of size n. For example $F = X^5Y^3 + 2X^4Y^4 + X^3Y^5 = X^3Y^3(X+Y)^2$ has the pattern $(3, 3, 2)$, illustrated by the Young diagram



$l_1 = 3$
$l_2 = 3$
$l_3 = 2$

4 Some Stabilizers in Projective 1-Space

The matrix g as in (1) transforms Y to $(-cX + aY)/\delta$. Hence it transforms the line kY to itself if and only if $c = 0$. Thus the stabilizer of the corresponding point $[Y] \in \mathbb{P}^1$ in $G = GL_2$ is

$$G_{[Y]} = B.$$

In the same way

$$G_{[X]} = B^-.$$

Or more generally:

Proposition 4 *The stabilizer in GL_2 of a single point of $\mathbb{P}^1$ is a Borel subgroup, conjugated with B.*

For pairs of different points we get

$$G_{([X],[Y])} = G_{[X]} \cap G_{[Y]} = B \cap B^- = T.$$

or more generally:

Proposition 5 *The (pointwise) stabilizer in GL_2 of a pair of different points in $\mathbb{P}^1$ is a maximal torus, conjugated with T.*

If we consider a *set* of two points the stabilizer is somewhat larger: Beside matrices that fix both points we also have to consider matrices that interchange them. Clearly the matrix I interchanges $[X]$ and $[Y]$, hence stabilizes the set $\{[X],[Y]\}$. An arbitrary matrix g that interchanges $[X]$ and $[Y]$ transforms X to λY and Y to μX with $\lambda, \mu \in k^\times$. Thus

$$\lambda Y = g \cdot X = \frac{d}{\delta} X - \frac{b}{\delta} Y \quad \text{and} \quad \mu X = g \cdot Y = -\frac{c}{\delta} X + \frac{a}{\delta} Y,$$

enforcing $a = d = 0$, hence g is in the coset $IT \subseteq GL_2$, hence in N.

Proposition 6 *The stabilizer in GL_2 of a two-element subset $\{x, y\} \subseteq \mathbb{P}^1$ is conjugated with N.*

In this way we get exact sequences of group homomorphisms and commutative squares (where $\mathcal{S}_r$ is the symmetric group on r elements $\{1, \ldots, r\}$):

$$
\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & G_{(x,y)} & \xrightarrow{\subseteq} & G_{\{x,y\}} & \longrightarrow & \mathcal{S}_2 & \longrightarrow & \mathbf{1} \\
 & & \cong\updownarrow & & \cong\updownarrow & & =\updownarrow & & \\
\mathbf{1} & \longrightarrow & T & \xrightarrow{\subseteq} & N & \longrightarrow & \mathcal{S}_2 & \longrightarrow & \mathbf{1}
\end{array}
$$

Next we consider triples of different points of $\mathbb{P}^1$. By Proposition 3 each such triple is in the GL_2-orbit of $([X],[Y],[X+Y])$. If $g \in GL_2$ fixes this special triple pointwise, it must be in T by Proposition 5, hence of the form $D(s,t)$ with s, $t \in k^\times$. Moreover

$$D(s,t)\cdot(X+Y) = \frac{t}{st}X + \frac{s}{st}Y = \frac{1}{s}X + \frac{1}{t}Y$$

is a multiple of $X+Y$ if and only if $s=t$. Hence the stabilizer is Z—note that Z acts trivially on $\mathbb{P}^1$.

Proposition 7 *The (pointwise) stabilizer in GL_2 of a triple of different points in $\mathbb{P}^1$ is Z, the center of GL_2.*

In particular this result implies that the action of $PGL_2 = GL_2/Z$ on $\mathbb{P}^1$ is sharply 3-transitive:

Corollary 4 *If $(x,y,z) \in W$, then there is exactly one element $g \in PGL_2$ such that $x = g\cdot[X]$, $y = g\cdot[Y]$, and $z = g\cdot[X+Y]$.*

For an m-element subset $M = \{x_1,\ldots,x_m\} \subseteq \mathbb{P}^1$ with $m \geq 3$ Proposition 7 implies that the pointwise stabilizer is Z. Thus we get a sequence:

$$\mathbf{1} \longrightarrow Z \xrightarrow{\subseteq} G_M \xrightarrow{\Phi} \mathcal{S}_m \dashrightarrow \mathbf{1}$$

The rightmost arrow is dashed since we don't know whether the sequence is exact at $\mathcal{S}_m$, i.e. whether Φ is surjective. We are going to prove this in the case $m=3$. In the general case we only have:

Corollary 5 *Let $M = \{x_1,\ldots,x_m\} \subseteq \mathbb{P}^1$ be an m-element subset with $m \geq 3$. Then the stabilizer of M in GL_2 is an extension of order $\leq m!$ of the center Z. The stabilizers of M in SL_2 and in PGL_2 are finite.*

In the special case of a three-element subset $M = \{x,y,z\} \subseteq \mathbb{P}^1$ we get a diagram where H is the stabilizer of the set $\{[X],[Y],[X+Y]\}$:

$$
\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & G_{(x,y,z)} & \xrightarrow{\subseteq} & G_M & \xrightarrow{\Phi} & \mathcal{S}_3 & \dashrightarrow & \mathbf{1} \\
 & & =\updownarrow & & \cong\updownarrow & & =\updownarrow & & \\
\mathbf{1} & \longrightarrow & Z & \xrightarrow{\subseteq} & H & \xrightarrow{\Phi'} & \mathcal{S}_3 & \dashrightarrow & \mathbf{1}
\end{array}
$$

Since the matrix I interchanges the linear forms X and Y, it fixes $X+Y$. Therefore the image of Φ' contains the transposition (12). Now we consider the matrix

$$J = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with } J^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and } J^3 = -\mathbf{1}$$

that transforms

$$X \mapsto -Y, \quad Y \mapsto X+Y, \quad X+Y \mapsto X,$$

hence permutes the set $\{[X],[Y],[X+Y]\}$ cyclically. Thus the image of Φ' also contains the 3-cycle (123), and therefore is the whole of $\mathcal{S}_3$. We have proved:

Proposition 8 *The stabilizer in GL_2 of a three-element subset $M = \{x,y,z\}$ of $\mathbb{P}^1$ is an extension of the center Z of order 6 and maps to the full symmetric group $\mathcal{S}_3$ in a natural way.*

The group H (the case of $M = \{[X],[Y],[X+Y]\}$) is generated by the subgroup Z together with the matrices I and J.

5 Some Stabilizers of Binary Forms

Consider a binary form $F \in R_n$, and let $(l_1,\ldots,l_r)$ be the pattern of its factorization $F = L_1^{l_1}\cdots L_r^{l_r}$ into pairwise non-proportional linear froms L_i with $l_1 \geq \ldots \geq l_r > 0$. For $g \in G_F$, the stabilizer of F in the group $G = GL_2$, we have

$$F = g\cdot F = (g\cdot L_1)^{l_1}\cdots(g\cdot L_r)^{l_r}$$

with linear factors $g\cdot L_i$. Since the prime decomposition is unique we conclude that

$$g\cdot L_i \in k\,L_j$$

where j is an index with $l_j = l_i$. In other words, G_F permutes the linear factors of the same multiplicity. If F has a single linear factor L of multiplicity l, then necessarily $g\cdot L = cL$ with some $c \in k^\times$. In the general case we collect identical multipliers:

$$m_1 = l_1 = \ldots = l_{s_1} > m_2 = l_{s_1+1} = \ldots = l_{s_1+s_2} > \ldots > m_t = \ldots = l_r > 0$$

with $r = s_1 + \cdots + s_t$. Then we have an induced group homomorphism

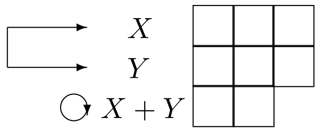
$$\Phi : G_F \longrightarrow \prod_{j=1}^{t} \mathcal{S}_{s_j}$$

into a product of symmetric groups. Its kernel is

$$\ker\Phi = \{g \in G_F \mid g\cdot L_i \in k\,L_i \text{ for all } i = 1,\ldots,r\} = G_w \cap G_F \subseteq G_w$$

where $w \in (\mathbb{P}^1)^r$ is the r-tuple $([L_1],\ldots,[L_r])$.

In the example $F = X^5Y^3 + 2X^4Y^4 + X^3Y^5 = X^3Y^3(X+Y)^2$ with pattern $(3,3,2)$ the stabilizer G_F permutes $\{[L_1],[L_2]\}$ and fixes $[L_3]$. This is illustrated by the Young diagram



$l_1 = 3$, $l_2 = 3$, $l_3 = 2$; $m_1 = l_1 = l_2 = 3$, $m_2 = l_3 = 2$

In the case $r \geq 3$ we know from Proposition 7 that $G_w = Z$.

Lemma 1 *Let n be the degree of the binary form F. Then $Z \cap G_F = Z_n$.*

Proof. The group $G_w = Z$ consists of the scalar matrices $g = c\mathbf{1}$ with $c \in k^\times$. Since $g \cdot F = c^{-n}F$, the matrix $c\mathbf{1}$ stabilizes F if and only if c is an n^{th} root of 1, that is $g \in Z_n$. $\diamond$

Thus we have proved statement (iii) of the following theorem:

Theorem 1 *Let $F \in R_n$ be a binary form, and r be the number of its pairwise non-proportional linear factors, $r' = \min\{3, r\}$. Let H be the stabilizer of F in GL_2. Then $\dim H = 3 - r'$. More precisely:*

(i) *If $r = 1$, then H is conjugated with the group of matrices $\begin{pmatrix} a & b \\ 0 & \varepsilon \end{pmatrix}$ where $a \in k^\times$, $b \in k$, and ε is an n^{th} root of 1.*

(ii) *If $r = 2$, then H is a finite extension of a one-dimensional torus.*

(iii) *If $r \geq 3$, then the H is finite.*

Proof. (i) By Corollary 1 in Section 3 we may assume that $F = Y^n$. Let $g \in H$. Then g stabilizes $[Y]$. By Proposition 4 we have $g \in B$, $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Then

$$g \cdot Y^n = \left(\frac{a}{ad}\right)^n Y^n = \frac{1}{d^n} Y^n,$$

implying that $d^n = 1$.

(ii) By Corollary 2 in Section 3 we may assume that $F = X^{n-r}Y^r$ for some $r \in \{1, \ldots, n-1\}$ with $r \geq n/2$. Let $g \in H$. Then g stabilizes the set $\{[X],[Y]\}$.

First assume that $r = n/2$. Then $g \in N$ by Proposition 6, $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ or $g = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. In the first case $\delta = ad$ and $g \cdot F = (1/a^r d^r)F$, thus ad is an r^{th} root of 1. In the second case $\delta = -bc$ and $g \cdot F = (1/b^r c^r)F$, thus bc is an r^{th} root of 1. In summary, g has the form

$$g = \begin{pmatrix} a & 0 \\ 0 & \eta/a \end{pmatrix} = D(a, 1/a)D(1, \eta) \quad \text{or} \quad \begin{pmatrix} 0 & b \\ \eta/b & 0 \end{pmatrix} = D(b, 1/b)D(1, \eta)I$$

where η is an r^{th} root of 1. Thus H is an extension of order $2r'$ of the one-dimensional torus $T' = T \cap SL_2 = \{D(t, 1/t) \,|\, t \in k^\times\}$ where r' is the number of r^{th} roots of 1 in k.

Now assume that $r > n/2$. Then g stabilizes the pair $([X], [Y])$ pointwise, hence $g \in T$ by Proposition 5, $g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $\delta = ad$. From $g \cdot F = 1/(a^{n-r}d^r)F$, we conclude that $a^{n-r}d^r = 1$. Therefore H is the kernel of the surjective homomorphism

$$\psi \colon T \longrightarrow k^\times, \quad D(a, d) \mapsto a^{n-r}d^r$$

and, by the way, contains the image of the one-parameter subgroup

$$\lambda \colon k^\times \longrightarrow T, \quad t \mapsto D(t^r, t^{r-n}).$$

Hence H has dimension 1 and therefore satisfies the assertion of (ii).

(iii) See the preliminary remarks. ◇

Corollary 6 *Let $F \in R_n$ be a binary form, and r be the number of its pairwise non-proportional linear factors. Let H' be the stabilizer of F in SL_2.*

(i) *If $r = 1$, then H' is a finite extension of a maximal unipotent subgroup of SL_2 of order n', the number of n^{th} roots of 1 in k.*

(ii) *If $r = 2$ and both linear factors have multiplicity $n/2$, then H' is a maximal torus of SL_2 if r is odd and* $\operatorname{char} k \neq 2$, *an extension of order 2 if r is even or* $\operatorname{char} k = 2$ *(namely a Cartan subgtoup of SL_2).*

(iii) *If $r = 2$ and the two linear factors have different multiplicities, then H' is finite.*

(iv) *If $r \geq 3$, then the H' is finite.*

Proof. Since we restrict the action from GL_2 to SL_2, the orbit representatives used in the proof of Theorem 1 hold only up to scalar factors. These factors however don't affect the stabilizers. Thus we only have to intersect $H' = H \cap SL_2$ (for the representatives of the GL_2-orbits).

(i) The condition $g \in SL_2$ enforces $a = 1/\varepsilon$. Hence H consists of the matrices

$$\begin{pmatrix} 1/\varepsilon & b \\ 0 & \varepsilon \end{pmatrix} \quad \text{with } b \in k \text{ and } \varepsilon \in k^\times \text{ an } n^{\text{th}} \text{ root of unity.}$$

(ii) In the proof of (ii) of the theorem H' consists of the matrices $D(a, 1/a)D(1, \eta)$ with $\eta = 1$ and $D(b, 1/b)D(1, \eta)I$ with $\eta = -1$ (if n is odd) or $\eta = 1$ (if $\operatorname{char} k = 2$).

(iii) A diagonal matrix $D(a, 1/a)$ stabilizes $X^{n-r}Y^r$ if and only if $a^{n-2r} = 1$. Therefore H' is finite.

(iv) immediate since even H is finite. ◇

Corollary 7 *If $F \in R_n$ has no linear factor of multiplicity $\geq n/2$, then the stabilizer of F in GL_2 is finite.*

Proof. The assumption implies that F has at least three pairwise non-proportional linear factors. ◇