Identities for Binomial Coefficients and Pascal Tableaus

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Figures 1 and 2 illustrate two summation formulas for binomial coefficients that will be proved and analyzed in a rather general form in the following.

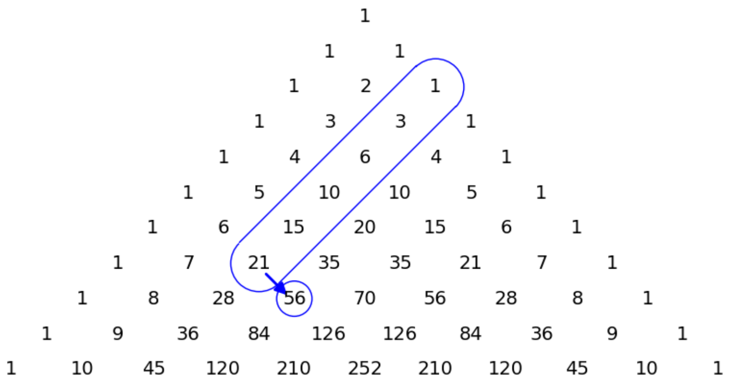

Figure 1: Pascal's triangle with the relation $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}+\binom{7}{2}=\binom{8}{3}$

1 Pascal Tableaus

Definition Let M be a $\mathbb{Z}$-module (most applications use $M=\mathbb{Z}$). A **Pascal tableau** in M is a map

$$T\colon \mathbb{N}\times\mathbb{N} \longrightarrow M$$

with the following properties, see Figure 3:

(i) $T(m,0)\in M$ arbitrary for all $m\in\mathbb{N}$.

(ii) $T(m,n)=0$ for all $n>m$.

(iii) $T(n,n)=T(0,0)$ for all $n\geq 1$.

(iv) For $m>n\geq 1$

$$T(m,n)=T(m-1,n-1)+T(m-1,n).$$

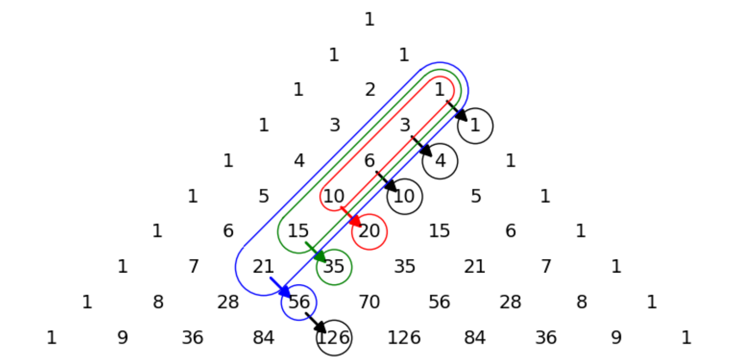

Figure 2: Pascal's triangle with the relation $6\cdot\binom{2}{2}+5\cdot\binom{3}{2}+4\cdot\binom{4}{2}+3\cdot\binom{5}{2}+2\cdot\binom{6}{2}+\binom{7}{2}=\binom{9}{4}$

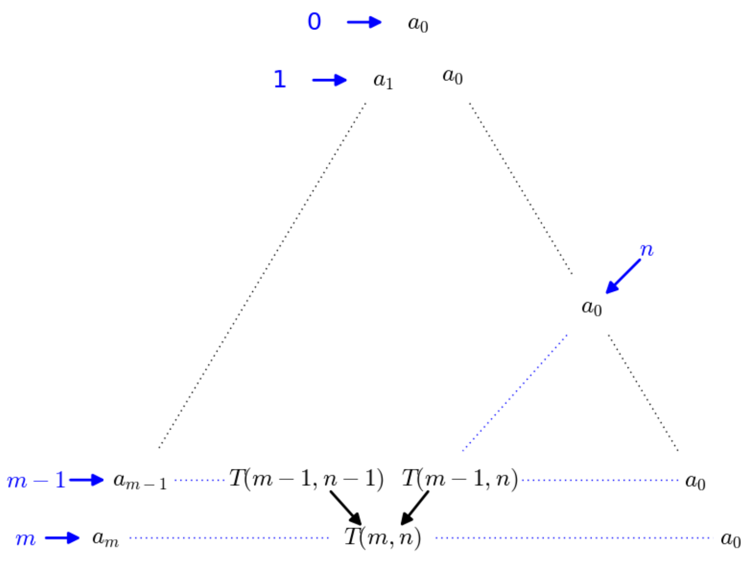

Figure 3: A general Pascal tableau

Example $M = \mathbb{Z}$, $T(m,n) = \binom{m}{n}$.

In this general context the rule illustrated by Figure 1 looks as follows:

Lemma 1 *Let T be a Pascal tableau in M. Then for integers $m \geq n \geq 1$:*

$$\sum_{k=n}^{m} T(k,n) = T(m+1, n+1)\,.$$

Proof. Induction on $m = n, n+1, \ldots$ with fixed n. The base case $m = n$ is obvious: The lefthand side is $T(n,n) = T(0,0)$, and the righthand side, $T(n+1,n+1) = T(0,0)$.

Now let $m \geq n+1$. Then by induction

$$\sum_{k=n}^{m} T(k,n) = T(m,n) + \underbrace{\sum_{k=n}^{m-1} T(k,n)}_{=T(m,n+1)} = T(m+1,n+1).$$

$\diamond$

This method of proof yields more, as Figure 2 suggests:

Proposition 1 *Let T be a Pascal tableau in M. Then for integers $m \geq n \geq 1$, $r \geq 0$:*

$$\sum_{k=n}^{m} \binom{m-k+r}{r} T(k,n) = T(m+r+1, n+r+1)\,.$$

Proof. Double induction on r and m. For $r = 0$ the assertion is proved by Lemma 1. For $m = n$ (with arbitrary r) it is trivial: The lefthand side is $\binom{r}{r} T(n,n) = T(0,0)$, the righthand side, $T(n+r+1, n+r+1) = T(0,0)$.

For the inductive step let $m \geq n+1$ and $r \geq 1$. The sum decomposes as

$$\sum_{k=n}^{m} \binom{m-k+r}{r} T(k,n) = \sum_{k=n}^{m} \left[\binom{m-k+r-1}{r-1} + \binom{m-k+r-1}{r}\right] T(k,n).$$

Separate evaluation of the two summands yields:

$$\sum_{k=n}^{m} \binom{m-k+r-1}{r-1} T(k,n) = T(m+r, n+r)$$

by induction on r,

$$\begin{aligned}\sum_{k=n}^{m} \binom{m-k+r-1}{r} T(k,n) &= \sum_{k=n}^{m-1} \binom{m-k+r-1}{r} T(k,n) \\ &= \sum_{k=n}^{q} \binom{q-k+r}{r} T(k,n) = T(q+r+1, n+r+1)\end{aligned}$$

by induction on m (since $q = m - 1$). The complete sum is

$$T(m+r, n+r) + T(m+r, n+r+1) = T(m+r+1, n+r+1)$$

by the defining rule of a Pascal tableau. $\diamond$

The special case $r = 0$ is in Lemma 1, the cases $r = 1$ and $r = 2$ in explicit form look like this:

Corollary 1 *Let T be a Pascal tableau, and $m \geq n \geq 1$. Then:*

$$\text{(1)} \qquad T(m+2, n+2) = \sum_{k=n}^{m} (m-k+1)\, T(k,n),$$

$$\text{(2)} \qquad T(m+3, n+3) = \sum_{k=n}^{m} \frac{(m-k+1)(m-k+2)}{2}\, T(k,n).$$

Another interesting special case of Proposition 1 is $n = 0$. Then the formula becomes

$$\sum_{k=0}^{m} \binom{m-k+r}{r} T(k,0) = T(m+r+1, r+1)\,.$$

Setting $q = m + r + 1$, $n = r + 1$ (note the changed meaning of n), hence $m = q - n$, the formula transforms to

$$\sum_{k=0}^{q-n} \binom{q-1-k}{n-1} T(k,0) = T(q,n)\,.$$

Changing the meaning of m and denoting q by m results in a formula that expresses the general term of a Pascal tableau by its first column:

Corollary 2 *Let T be a Pascal tableau, and $m \geq n \geq 1$. Then*

$$T(m,n) = \sum_{k=0}^{m-n} \binom{m-1-k}{n-1} T(k,0)\,.$$

Applying Proposition 1 to the binomial coefficients $T(m,n) = \binom{m}{n}$ yields the formula

$$\sum_{k=n}^{m} \binom{m-k+r}{r} \binom{k}{n} = \binom{m+r+1}{n+r+1}.$$

Setting $N = m + 1$ and $q = n + 1$ yields the variant

$$\binom{N+r}{q+r} = \sum_{k=q-1}^{N-1} \binom{N-1-k+r}{r} \binom{k}{q-1} = \sum_{i=1}^{N-q+1} \binom{i-1+r}{r} \binom{N-i}{q-1},$$

thus, once more renaming the variables:

Corollary 3 *For integers $N \geq n \geq 1$, $r \geq 0$:*

$$\binom{N+r}{n+r} = \sum_{i=1}^{N-n+1} \binom{i+r-1}{r}\binom{N-i}{n-1}.$$

The explicit form of the special cases $r = 0, 1, 2$ is:

Corollary 4 *For integers $N \geq n \geq 1$:*

$$\text{(1)} \qquad \binom{N}{n} = \sum_{i=1}^{N-n+1} \binom{N-i}{n-1},$$

$$\text{(2)} \qquad \binom{N+1}{n+1} = \sum_{i=1}^{N-n+1} i \cdot \binom{N-i}{n-1},$$

$$\text{(3)} \qquad \binom{N+2}{n+2} = \sum_{i=1}^{N-n+1} \frac{i\,(i+1)}{2} \cdot \binom{N-i}{n-1}.$$

We use the relation $i\,(i+1) = i^2 + i$, or $i^2 = 2 \cdot \frac{i\,(i+1)}{2} - i$, to slightly modify Formula (iii):

$$\begin{aligned}
\sum_{i=1}^{N-n+1} i^2 \cdot \binom{N-i}{n-1} &= 2 \cdot \binom{N+2}{n+2} - \binom{N+1}{n+1} \\
&= 2 \cdot \binom{N+1}{n+1} + 2 \cdot \binom{N+1}{n+2} - \binom{N+1}{n+1} \\
&= \binom{N+2}{n+2} + \binom{N+1}{n+2},
\end{aligned}$$

with the result:

Corollary 5 *For integers $N \geq n \geq 1$:*

$$\sum_{i=1}^{N-n+1} i^2 \cdot \binom{N-i}{n-1} = \binom{N+2}{n+2} + \binom{N+1}{n+2}.$$

In the same way Corollary 1 yields the more general result:

Corollary 6 *Let T be a Pascal tableau and $m \geq n \geq 1$. Then:*

$$\begin{aligned}
\sum_{k=n}^{m} (m-k+1)^2\, T(k,n) &= 2\,T(m+3,n+3) - T(m+2,n+2) \\
&= T(m+3,n+3) + T(m+2,n+3)
\end{aligned}$$

2 Sum and Difference Sequences

In this section M continues to be a $\mathbb{Z}$-module. We consider sequences $a = (a_n)_{n\in\mathbb{N}} = (a_0, a_1, a_2, \ldots)$ in M. They form the set $M^{\mathbb{N}}$ that is itself a $\mathbb{Z}$-module.

Definition Let $a \in M^{\mathbb{N}}$ be a sequence. The **sum sequence** $b \in M^{\mathbb{N}}$ of a is defined by

$$b_n = \sum_{i=0}^{n} a_i \quad \text{for all } n \in \mathbb{N},$$

the **difference sequence** $d \in M^{\mathbb{N}}$ by

$$d_n = a_n - a_{n-1} \quad \text{for all } n \geq 1, \quad \text{and } d_0 = a_0.$$

Obviously the difference sequence of the sum sequence is a itself, as is the sum sequence of the difference sequence. Thus we have two operators on sequences,

$$\sigma \colon M^{\mathbb{N}} \longrightarrow M^{\mathbb{N}} \quad (\text{“sum”}) \quad \text{and} \quad \delta \colon M^{\mathbb{N}} \longrightarrow M^{\mathbb{N}} \quad (\text{“difference”})$$

that are inverse to each other. For $a \in M^{\mathbb{N}}$ we use the notation $a^{(k)} := \sigma^k(a)$ for the k-fold sum operator.

Proposition 2 *Let $a \in M^{\mathbb{N}}$ be a sequence. Then the map*

$$T \colon \mathbb{N} \times \mathbb{N} \longrightarrow M, \quad T(m,n) := a_{m-n}^{(n)}$$

(and $T(m,n) = 0$ for $n > m$) is a Pascal tableau.

Proof. We have $T(n,n) = a_0$ for all $n \in \mathbb{N}$. And for $m > n \geq 1$

$$\begin{aligned} T(m-1,n-1) + T(m-1,n) &= a_{m-n}^{(n-1)} + a_{m-n-1}^{(n)} \\ &= a_{m-n}^{(n-1)} + \sum_{j=0}^{m-n-1} a_j^{(n-1)} = \sum_{j=0}^{m-n} a_j^{(n-1)} \\ &= a_{m-n}^{(n)} = T(m,n). \end{aligned}$$

◇

On the other hand, given a Pascal tableau T, consider the sequence a defined by $a_n = T(n,0)$. Then T is the Pascal tableau corresponding to a since Lemma 1 says that the sequence $T(*, n+1)$ is the sum sequence of $T(*, n)$. Thus we have a map

$$\Lambda \colon M^{\mathbb{N}} \longrightarrow M^{\mathbb{N}\times\mathbb{N}}$$

that maps the sequences bijectively to the set of Pascal tableaus.

In a more informal way we get a Pascal tableau from a sequence a by the following procedure:

- Write the sequence a as a column.
- For $k \geq 1$ construct column k from column $k-1$ as its sum sequence.
- Rotate this scheme by 45 degrees to the right to get the usual triangle shape.

For the constant sequence with value 1 the intermediate matrix is

$a^{(0)}$	$a^{(1)}$	$a^{(2)}$	$a^{(3)}$	$a^{(4)}$	$a^{(5)}$	$a^{(6)}$	$a^{(7)}$	$\dots$
1	1	1	1	1	1	1	1	
1	2	3	4	5	6	7	8	
1	3	6	10	15	21	28	36	
1	4	10	20	35	56	84	120	
1	5	15	35	70	126	210	330	
1	6	21	56	126	252	462	792	
1	7	28	84	210	462	924	1716	
1	8	36	120	330	792	1716	3432	
$\vdots$								$\ddots$

We recover the original Pascal triangle.

From Propositions 1 and 2 we immediately conclude:

Corollary 1 *Let $a \in M^{\mathbb{N}}$ be a sequence. Then for all integers $n, q, r \geq 0$*

$$\sum_{i=0}^{q} \binom{q+r-i}{r} a_i^{(n)} = a_q^{(n+r+1)}.$$

And Corollary 2 of Proposition 1 yields an explicit expression of the tableau entries by the generating series and binomial coefficients:

Corollary 2 *Let $a \in M^{\mathbb{N}}$ be a sequence and T be the corresponding Pascal tableau. Then for $m \geq n \geq 1$*

$$T(m,n) = \sum_{k=0}^{m-n} \binom{m-1-k}{n-1} a_k\,.$$