The Clebsch-Gordan Isomorphism

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1 Some Differential Calculus

Let k be a commutative ring with 1. (However the main results require k to be a field of characteristic 0.) Let $R = k[X]$ be the polynomial ring in the indeterminates $X = (X_1, \ldots, X_n)$. The ring R is graded by the degree of a polynomial:

$$R = \bigoplus_{d \in \mathbb{N}} R_d \,.$$

Let $Y = (Y_1, \ldots, Y_n)$ be another set of indeterminates and $S = k[X, Y]$ the extended polynomial ring. It is bigraded by the degrees in X and Y:

$$S = \bigoplus_{d,e \in \mathbb{N}} S_{de} \,.$$

In a natural way $S_{d0} = R_d$.

We consider the derivation

$$D\colon S \longrightarrow S, \quad DF = Y_1 \partial_1 F + \cdots + Y_n \partial_n F = \sum_{\nu=1}^{n} Y_\nu \partial_\nu F,$$

where ∂_ν is the derivation by X_ν. The effect of D on a monomial

$$F = X^\alpha Y^\beta = X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_n^{\beta_n} \tag{1}$$

is

$$DF = \sum_{\nu=1}^{n} \alpha_\nu \frac{Y_\nu}{X_\nu} F.$$

This expression seems to be in the quotient ring $S[1/X_1, \ldots, 1/X_n]$, however the denominators cancel out except in the term with coefficient $\alpha_\nu = 0$. The effect of D is “replace one factor X_ν by $\alpha_\nu Y_\nu$, for each ν.”

Sometimes (for $d \geq 1$) we use an explicit denomination for the restriction

$$D_{de} = D_{|S_{de}} \colon S_{de} \longrightarrow S_{d-1,e+1},$$

in particular

$$D_{d0} \colon R_d \longrightarrow S_{d-1,1}.$$

Remark We'll occasionally encounter the simpler differential operator $R \longrightarrow R$, $f \mapsto X_1\partial_1 f + \cdots + X_n\partial_n f$. Its effect on the homogeneous part R_d is simply multiplication by the integer d, as is easily seen by applying it to the monomials X^α. (See also Proposition 2.)

The group $G = GL_n(k)$ of invertible $n \times n$-matrices acts on the variables X and Y separately. The actions on S_{10} and S_{01} are linear and correspond to the contragredient action of the natural action on k^n. These actions of G extend to S, resulting in a group of k-algebra automorphisms.

Proposition 1 *The derivation D is G-equivariant. In other words*

$$D(g \cdot F) = g \cdot DF$$

for all $g \in G$ and $F \in S$.

Proof. Let the linear action of $g \in G$ on the indeterminates be given be the equations

$$g \cdot X_i = \sum_{j=1}^n a_{ij} X_j \quad \text{and } g \cdot Y_i = \sum_{j=1}^n a_{ij} Y_j$$

Since D is k-linear it suffices to prove the assertion for monomials. So let F be given by equation (1). Then

$$g \cdot F = (g \cdot X_1)^{\alpha_1} \cdots (g \cdot X_n)^{\alpha_n} (g \cdot Y_1)^{\beta_1} \cdots (g \cdot Y_n)^{\beta_n},$$

$$\begin{aligned}
D(g \cdot F) &= \sum_{\nu=1}^n Y_\nu \partial_\nu \left(\prod_{i=1}^n (g \cdot X_i)^{\alpha_i} \right) \prod_{s=1}^n (g \cdot Y_s)^{\beta_s} \\
&= \sum_{\nu=1}^n Y_\nu \left[\sum_{i=1}^n \alpha_i (g \cdot X_i)^{\alpha_i - 1} a_{i\nu} \prod_{t \neq i} (g \cdot X_t)^{\alpha_t} \right] \prod_{s=1}^n (g \cdot Y_s)^{\beta_s} \\
&= \prod_{t=1}^n (g \cdot X_t)^{\alpha_t} \prod_{s=1}^n (g \cdot Y_s)^{\beta_s} \left[\sum_{\nu=1}^n \sum_{i=1}^n \alpha_i a_{i\nu} \frac{Y_\nu}{g \cdot X_i} \right]
\end{aligned}$$

$$\begin{aligned}
&= (g \cdot F) \left[\sum_{i=1}^n \alpha_i \frac{1}{g \cdot X_i} \sum_{\nu=1}^n a_{i\nu} Y_\nu \right] \\
&= (g \cdot F) \left[\sum_{i=1}^n \alpha_i \frac{g \cdot Y_i}{g \cdot X_i} \right] = g \cdot \left(F \sum_{i=1}^n \alpha_i \frac{Y_i}{X_i} \right) = g \cdot DF.
\end{aligned}$$

◇

The substitution $Y \mapsto X$ defines a G-equivariant k-algebra homomorphism $\sigma\colon S \longrightarrow R$ with $\sigma(S_{de}) = R_{d+e}$. For a monomial $F = X^\alpha Y^\beta \in S_{de}$ (with $\alpha_1 + \cdots + \alpha_n = d$, $\beta_1 + \cdots + \beta_n = e$) we have

$$(\sigma \circ D)\, F = \sum_{\nu=1}^{n} X_\nu (\partial_\nu F)(X, X) = \sum_{\nu=1}^{n} \alpha_\nu F(X, X) = d\, F(X, X)$$

(where d denotes the integer, not a differential). Since D and σ both are linear this equality holds for arbitrary $F \in S_{de}$:

Proposition 2 *$\sigma \circ D = d\,\sigma$ on S_{de} for all d, $e \in \mathbb{N}$.*

In other words the following diagram commutes:

$$\begin{array}{ccc}
S_{de} & \xrightarrow{\;D\;} & S_{d-1,e+1} \\
\sigma \downarrow & & \downarrow \sigma \\
R_{d+e} & \xrightarrow[d \cdot \mathbf{1}]{} & R_{d+e}
\end{array}$$

Corollary 1 *$\sigma \circ D^i = \frac{d!}{(d-i)!}\,\sigma$ on S_{de} for $0 \le i \le d$ and all $e \in \mathbb{N}$.*

Proof. The proof consists of the following commutative diagram: ◇

$$\begin{array}{ccccccccc}
S_{de} & \xrightarrow{\;D\;} & S_{d-1,e+1} & \xrightarrow{\;D\;} & \cdots & \xrightarrow{\;D\;} & S_{d-i,e+i} \\
\sigma \downarrow & & \downarrow \sigma & & & & \downarrow \sigma \\
R_{d+e} & \xrightarrow[d \cdot \mathbf{1}]{} & R_{d+e} & \xrightarrow[(d-1) \cdot \mathbf{1}]{} & \cdots & \xrightarrow[(d-i+1) \cdot \mathbf{1}]{} & R_{d+e}
\end{array}$$

2 The Clebsch-Gordan Isomorphism

Assume $n = 2$. Thus $G = GL_2(k)$ consists of the 2×2-matrices whose determinant is invertible in k. We use the distinguished polynomial

$$\Delta = X_1 Y_2 - X_2 Y_1 \in S_{11}.$$

Lemma 1 *For $g \in GL_2(k)$ we have $g \cdot \Delta = \frac{1}{\det g}\,\Delta$, in other words, Δ is a relative invariant of weight -1.*

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k)$ with determinant $\delta = \det g = ad - bc \in k^\times$, the multiplicative group of invertible elements of k. The explicit formulas for the action on the indeterminates are

$$g \cdot X_1 = \frac{d}{\delta} X_1 - \frac{b}{\delta} X_2, \quad g \cdot X_2 = -\frac{c}{\delta} X_1 + \frac{a}{\delta} X_2,$$

and analogously for Y_1, Y_2. Then

$$\begin{aligned} g \cdot \Delta &= (g \cdot X_1)(g \cdot Y_2) - (g \cdot X_2)(g \cdot Y_1) \\ &= \frac{1}{\delta^2} [(dX_1 - bX_2)(-cY_1 + aY_2) - (-cX_1 + aX_2)(dY_1 - bY_2)] \\ &= \frac{1}{\delta^2} [0X_1Y_1 + \delta X_1Y_2 - \delta X_2Y_1 + 0X_2Y_2] \\ &= \frac{1}{\delta} \Delta. \end{aligned}$$

◇

We assume $d \geq e$ and $0 \leq i \leq e$. For $f \in R_{d+e-2i}$ we have $D^{e-i} f \in S_{d-i,e-i}$. Multiplying by $\Delta^i \in S_{ii}$ we get

$$\Delta^i D^{e-i} f \in S_{de} \quad \text{for } i = 0, \ldots, e.$$

This defines linear maps

$$\varphi_i \colon R_{d+e-2i} \longrightarrow S_{de}, \quad f \mapsto \Delta^i D^{e-i} f.$$

Proposition 3 *The maps φ_i are relatively G-equivariant of weight i, that is*

$$\varphi_i(g \cdot f) = (\det g)^i \, g \cdot \varphi_i(f)$$

for all $g \in GL_2(k)$ and $f \in R_{d+e-2i}$.

Proof. For $g \in GL_2(k)$ with $\delta = \det g$ and $f \in R_{d+e-2i}$ we have

$$g \cdot \varphi_i(f) = (g \cdot \Delta^i)(g \cdot D^{e-i} f) = (\frac{1}{\delta^i} \Delta^i) D^{e-i}(g \cdot f) = \frac{1}{\delta^i} \varphi_i(g \cdot f)$$

by Lemma 1 and Proposition 1. ◇

Remark The maps φ_i seem to be artificial constructs. However they are equivariant for $SL_2(k)$ and thus embed some (irreducible if k is a field of characteristic 0) SL_2-modules of type R_j into the SL_2-module $S_{de} \cong R_d \otimes R_e$.

We combine the maps φ_i and get the Clebsch-Gordan map

$$\Phi\colon R_{d+e} \oplus R_{d+e-2} \oplus \cdots \oplus R_{d-e} \longrightarrow S_{de},$$

$$\Phi(f_0, \ldots, f_e) = \varphi_0(f_0) + \cdots + \varphi_e(f_e).$$

We know that Φ is linear and SL_2-equivariant.

Theorem 1 (Sylvester 1878) *Let k be a field of characteristic 0. Then the Clebsch-Gordan map Φ is an SL_2-equivariant isomorphism.*

Proof. (Springer [4]) The dimensions are equal:

$$\begin{aligned}\dim(R_{d+e} \oplus \cdots \oplus R_{d-e}) &= (d+e+1) + \cdots + (d-e+1)\\ &= \sum_{i=0}^{e}(d+e+1-2i) = (e+1)(d+e+1) - 2\sum_{i=0}^{e} i\\ &= (e+1)(d+e+1) - e(e+1)\\ &= (d+1)(e+1) = \dim S_{de}\end{aligned}$$

It only remains to prove that Φ is injective. Let $(f_0, \ldots, f_e) \in \ker\Phi$. Thus

$$0 = \sum_{i=0}^{e} \Delta^i D^{e-i} f_i\,. \tag{2}$$

The substitution homomorphism $\sigma : Y \mapsto X$ yields $\sigma\Delta^i = 0$ for $i \geq 1$, and we get

$$0 = (\sigma \circ D^e) f_0 = e!\, \sigma f_0 = e!\, f_0$$

after applying e times Proposition 2. This implies $f_0 = 0$.

Now in Equation (2) one factor Δ cancels out, leaving

$$0 = \sum_{i=1}^{e} \Delta^{i-1} D^{e-i} f_i\,.$$

The same reasoning shows that $(e-1)!\, f_1 = 0$, or $f_1 = 0$.

Proceeding by induction we conclude that all $f_i = 0$ for $i = 0, \ldots, e$. Thus the kernel of Φ contains 0 only. $\diamond$

The theorem describes (in characteristic 0) the decomposition of the SL_2-module $S_{de} \cong R_d \otimes R_e$ into irreducible components. In a more old-fashioned way it may be expressed as

Corollary 2 *Each $F \in S_{de}$ has a unique decomposition*

$$F(X,Y) = \sum_{i=0}^{e} \Delta(X,Y)^i (D^{e-i} f_i)(X,Y)$$

with $f_i \in R_{d+e-2i}$.

This expression involves the powers of the differential operator D. Here is a formula for their effect on monomials:

Proposition 4 *The power D^j of $D : S \longrightarrow S$ acts on the monomial $X_1^r X_2^s Y_1^t Y_2^u$ by the formula*

$$D^j(X_1^r X_2^s Y_1^t Y_2^u) = \sum_{\nu=0}^{j} \binom{j}{\nu} \frac{r!}{(r-j+\nu)!} \frac{s!}{(s-\nu)!} X_1^{r-j+\nu} X_2^{s-\nu} Y_1^{t+j-\nu} Y_2^{u+\nu}.$$

Proof. The formula is obviously true for $j = 0$. Proceeding by induction we assume that it is true for $j - 1$. Then

$$D^j(X_1^r X_2^s Y_1^t Y_2^u) = D(D^{j-1}(X_1^r X_2^s Y_1^t Y_2^u))$$

$$= D\left[\sum_{\nu=0}^{j-1} \binom{j-1}{\nu} \frac{r!}{(r-j+1+\nu)!} \frac{s!}{(s-\nu)!} X_1^{r-j+1+\nu} X_2^{s-\nu} Y_1^{t+j-1-\nu} Y_2^{u+\nu}\right]$$

$$= \sum_{\nu=0}^{j-1} \left[\binom{j-1}{\nu} (r-j+1+\nu) \frac{r!}{(r-j+1+\nu)!} \frac{s!}{(s-\nu)!} X_1^{r-j+\nu} X_2^{s-\nu} Y_1^{t+j-\nu} Y_2^{u+\nu}\right.$$

$$\left. + \binom{j-1}{\nu} (s-\nu) \frac{r!}{(r-j+1+\nu)!} \frac{s!}{(s-\nu)!} X_1^{r-j+1+\nu} X_2^{s-\nu-1} Y_1^{t+j-1-\nu} Y_2^{u+\nu+1}\right]$$

$$= \sum_{\nu=0}^{j-1} \binom{j-1}{\nu} \frac{r!}{(r-j+\nu)!} \frac{s!}{(s-\nu)!} X_1^{r-j+\nu} X_2^{s-\nu} Y_1^{t+j-\nu} Y_2^{u+\nu}$$

$$+ \sum_{\nu=1}^{j} \binom{j-1}{\nu-1} \frac{r!}{(r-j+\nu)!} \frac{s!}{(s-\nu)!} X_1^{r-j+\nu} X_2^{s-\nu} Y_1^{t+j-\nu} Y_2^{u+\nu}$$

$$= \sum_{\nu=0}^{j} \left[\binom{j-1}{\nu} + \binom{j-1}{\nu-1}\right] \frac{r!}{(r-j+\nu)!} \frac{s!}{(s-\nu)!} X_1^{r-j+\nu} X_2^{s-\nu} Y_1^{t+j-\nu} Y_2^{u+\nu}$$

$$= \sum_{\nu=0}^{j} \binom{j}{\nu} \frac{r!}{(r-j+\nu)!} \frac{s!}{(s-\nu)!} X_1^{r-j+\nu} X_2^{s-\nu} Y_1^{t+j-\nu} Y_2^{u+\nu}.$$

◇

3 Cayley's Ω-Operator

The f_i in Corollary 2 have an explicit description in terms of F using a suitable differential operator. The corresponding formula was already given by Cayley [1] in 1856, and proved by Sylvester [6], see Corollary 5 below.

We continue with $n = 2$ (although Ω makes sense for arbitrary n). We consider the ring $S = k[X, Y] = k[X_1, X_2, Y_1, Y_2]$ and denote the partial derivatives with respect to X_i by $\partial_i = \partial/\partial X_i$ and with respect to Y_i by $\tilde{\partial}_i = \partial/\partial Y_i$. Then we define the differential operator

$$\Omega\colon S \longrightarrow S \quad \text{as} \quad \Omega = \partial_1\tilde{\partial}_2 - \tilde{\partial}_1\partial_2.$$

Note that the partial derivatives commute. Obviously the operator Ω is k-linear, and $\Omega(S_{de}) \subseteq S_{d-1,e-1}$. Its effect on a product yields a somewhat obscure formula (that however in certain special situations will turn out as useful): Let $F_1, F_2 \in S$. Then

$$\begin{aligned}\Omega(F_1F_2) &= \partial_1\tilde{\partial}_2(F_1F_2) - \tilde{\partial}_1\partial_2(F_1F_2)\\ &= \partial_1(\tilde{\partial}_2(F_1)F_2 + F_1\tilde{\partial}_2(F_2)) - \tilde{\partial}_1(\partial_2(F_1)F_2 + F_1\partial_2(F_2))\\ &= \partial_1\tilde{\partial}_2(F_1)F_2 + \tilde{\partial}_2(F_1)\partial_1(F_2) + \partial_1(F_1)\tilde{\partial}_2(F_2) + F_1\partial_1\tilde{\partial}_2(F_2)\\ &\quad - \tilde{\partial}_1\partial_2(F_1)F_2 - \partial_2(F_1)\tilde{\partial}_1(F_2) - \tilde{\partial}_1(F_1)\partial_2(F_2) - F_1\tilde{\partial}_1\partial_2(F_2)\end{aligned}$$

Collecting similar terms we get the product rule for Ω, statement (i) of the following lemma:

Lemma 2 (i) *For $F_1, F_2 \in S$*

$$\Omega(F_1F_2) = \Omega(F_1)\,F_2 + F_1\,\Omega(F_2) + \begin{vmatrix}\partial_1F_1 & \tilde{\partial}_1F_2\\ \partial_2F_1 & \tilde{\partial}_2F_2\end{vmatrix} - \begin{vmatrix}\tilde{\partial}_1F_1 & \partial_1F_2\\ \tilde{\partial}_2F_1 & \partial_2F_2\end{vmatrix}$$

(ii) $\Omega(\Delta^i) = i\,(i+1)\,\Delta^{i-1} \in S_{i-1,i-1}$, *in particular* $\Omega(\Delta) = 2 \in S_{00} = k$.

(iii) *For $F \in S_{de}$*

$$\Omega(\Delta^iF) = i\,(d+e+i+1)\,\Delta^{i-1}F + \Delta^i\,\Omega(F)$$

(iv) *For $f \in R_d$*

$$\Omega(\Delta^if) = i\,(d+i+1)\,\Delta^{i-1}f$$

Proof. (ii) For $F_1 = \Delta^i$ we get (remember $\Delta = X_1Y_2 - X_2Y_1$)

$$\begin{aligned}\partial_1(F_1) &= iY_2\Delta^{i-1} \qquad & \tilde{\partial}_1(F_1) &= -iX_2\Delta^{i-1}\\ \partial_2(F_1) &= -iY_1\Delta^{i-1} \qquad & \tilde{\partial}_2(F_1) &= iX_1\Delta^{i-1}\end{aligned}$$

Hence

$$\begin{aligned}\Omega(F_1) &= \partial_1\tilde{\partial}_2(F_1) - \tilde{\partial}_1\partial_2(F_1) = \partial_1(iX_1\Delta^{i-1}) + \tilde{\partial}_1(iY_1\Delta^{i-1})\\ &= i\,\Delta^{i-1} + i\,(i-1)\,X_1Y_2\Delta^{i-2} + i\,\Delta^{i-1} - i\,(i-1)\,X_2Y_1\Delta^{i-2}\\ &= 2i\,\Delta^{i-1} + i\,(i-1)\,[X_1Y_2 - X_2Y_1]\,\Delta^{i-2} = i\,(i+1)\,\Delta^{i-1}\end{aligned}$$

(iii) For $F_1 = \Delta^i$ and $F_2 = F \in S_{de}$ we get

$$\begin{aligned}
\begin{vmatrix} \partial_1 F_1 & \tilde\partial_1 F_2 \\ \partial_2 F_1 & \tilde\partial_2 F_2 \end{vmatrix} - \begin{vmatrix} \tilde\partial_1 F_1 & \partial_1 F_2 \\ \tilde\partial_2 F_1 & \partial_2 F_2 \end{vmatrix} &= iY_2\Delta^{i-1}\tilde\partial_2 F + iY_1\Delta^{i-1}\tilde\partial_1 F \\
&\quad - (-iX_2\Delta^{i-1}\partial_2 F - iX_1\Delta^{i-1}\partial_1 F) \\
&= i\Delta^{i-1}[X_1\partial_1 + X_2\partial_2 + Y_1\tilde\partial_1 + Y_2\tilde\partial_2](F) \\
&= i\,(d+e)\,\Delta^{i-1}F
\end{aligned}$$

using the remark in Section 1. Combining (i) and (ii) yields

$$\begin{aligned}
\Omega(\Delta^i F) &= \Omega(\Delta^i)\,F + \Delta^i\,\Omega(F) + |\ldots| - |\ldots| \\
&= i\,(d+e+i+1)\,\Delta^{i-1}F + \Delta^i\,\Omega(F)
\end{aligned}$$

(iv) follows from (iii) setting $e = 0$ and using $\Omega(f) = 0$. $\diamond$

We are going to prove that Ω is relatively equivariant for the action of $G = GL_2(k)$. To this end we again consider an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with determinant $\delta = \det g = ad - bc$. Its effect on a polynomial $F \in k[X, Y]$ is

$$(g \cdot F)(X_1, X_2, Y_1, Y_2) = F(g \cdot X, g \cdot Y)$$

where

$$(g \cdot X, g \cdot Y) = (\frac{d}{\delta}X_1 - \frac{b}{\delta}X_2, -\frac{c}{\delta}X_1 + \frac{a}{\delta}X_2,\ \frac{d}{\delta}Y_1 - \frac{b}{\delta}Y_2, -\frac{c}{\delta}Y_1 + \frac{a}{\delta}Y_2)$$

This yields

$$\begin{aligned}
\tilde\partial_2(g \cdot F) &= \left[-\frac{b}{\delta}\tilde\partial_1 F + \frac{a}{\delta}\tilde\partial_2 F\right](g \cdot X, g \cdot Y) \\
\partial_2(g \cdot F) &= \left[-\frac{b}{\delta}\partial_1 F + \frac{a}{\delta}\partial_2 F\right](g \cdot X, g \cdot Y) \\
\partial_1\tilde\partial_2(g \cdot F) &= \left[-\frac{b}{\delta}\frac{d}{\delta}\partial_1\tilde\partial_1 F + \frac{b}{\delta}\frac{c}{\delta}\partial_2\tilde\partial_1 F + \frac{a}{\delta}\frac{d}{\delta}\partial_1\tilde\partial_2 F - \frac{a}{\delta}\frac{c}{\delta}\partial_2\tilde\partial_2 F\right](g \cdot X, g \cdot Y) \\
\tilde\partial_1\partial_2(g \cdot F) &= \left[-\frac{b}{\delta}\frac{d}{\delta}\tilde\partial_1\partial_1 F + \frac{b}{\delta}\frac{c}{\delta}\tilde\partial_2\partial_1 F + \frac{a}{\delta}\frac{d}{\delta}\tilde\partial_1\partial_2 F - \frac{a}{\delta}\frac{c}{\delta}\tilde\partial_2\partial_2 F\right](g \cdot X, g \cdot Y) \\
\Omega(g \cdot F) &= \left[\frac{ad}{\delta^2}\partial_1\tilde\partial_2 F - \frac{bc}{\delta^2}\tilde\partial_2\partial_1 F + \frac{bc}{\delta^2}\partial_2\tilde\partial_1 F - \frac{ad}{\delta^2}\tilde\partial_1\partial_2 F\right](g \cdot X, g \cdot Y) \\
&= \frac{1}{\delta}\,\Omega F(g \cdot X, g \cdot Y) \\
&= \frac{1}{\delta}\,g \cdot \Omega F.
\end{aligned}$$

This last equation may be stated as follows:

Proposition 5 *The operator Ω is relatively equivariant for the action of $G = GL_2(k)$ with weight -1.*

Corollary 3 *The operator Ω^i is relatively equivariant for the action of $G = GL_2(k)$ with weight $-i$.*

We again assume $d \geq e$. Then for each $i = 0, \ldots, e$ we have a pair of relatively G-equivariant linear maps

$$\varphi_i \colon R_{d+e-2i} \longrightarrow S_{de}, \quad f \mapsto \Delta^i D^{e-i} f$$

of weight i, and

$$S_{de} \xrightarrow{\Omega^i} S_{d-i,e-i} \xrightarrow{\sigma} R_{d+e-2i}$$

of weight $-i$. This suggests the questions: What is

- $\eta_i = (\sigma \circ \Omega^i) \circ \varphi_i$ on R_{d+e-2i} ?
- $\psi_i = \varphi_i \circ (\sigma \circ \Omega^i)$ on S_{de} ?

Note that η_i and ψ_i are G-equivariant.

Examples Let us start with the easy cases of η_i.

1. For $i = 0$ we have to consider $\eta_0 = \sigma \circ \Omega^0 \circ \varphi_0 = \sigma \circ D^e$ on $R_{d+e} \subseteq S_{d+e,0}$. This was calculated in Corollary 1 of Proposition 2: $\sigma \circ D^e = \frac{(d+e)!}{d!} \mathbf{1}$ on $S_{d+e,0}$. Hence

$$\eta_0(f) = \frac{(d+e)!}{d!} f \quad \text{for all } f \in R_{d+e}.$$

2. For $i = 1$ we have to consider $\eta_1 = \sigma \circ \Omega \circ \varphi_1$ on $R_{d+e-2} \subseteq S_{d+e-2,0}$.

$$R_{d+e-2} \xrightarrow{\varphi_1} S_{de} \xrightarrow{\Omega} S_{d-1,e-1} \xrightarrow{\sigma} R_{d+e-2}$$

where $\varphi_1(f) = \Delta\, D^{e-1} f$, and $D^{e-1} f \in S_{d-1,e-1}$. Applying Lemma 2 (iii) with $i = 1$ we get

$$\Omega(\varphi_1 f) = (d+e)\, D^{e-1} f + \Delta\, \Omega(D^{e-1} f).$$

Using $\sigma(\Delta) = 0$ and the formula for $\sigma \circ D^{e-1}$ from Corollary 1 of Proposition 2 this results in

$$\eta_1(f) = \sigma(\Omega(\varphi_1 f)) = (d+e)\, \sigma(D^{e-1} f) = (d+e)\, \frac{(d+e-2)!}{(d-1)!}\, f.$$

In both examples η_i is the identity map up to an integer factor. This observation generalizes to:

Theorem 2 *For all $d, e, i \in \mathbb{N}$, $0 \le i \le e \le d$,*

$$\eta_i = \gamma_{dei}\,\mathbf{1} \quad \text{on } R_{d+e-2i}$$

where $\gamma_{dei} \in \mathbb{Z}$ is given by the formula

$$\gamma_{dei} = \frac{i!}{(d-i)!}\,\frac{(d+e-i+1)!}{d+e-2i+1}\,.$$

The proof follows. Note that the coefficients γ_{dei} are integers, so the result is true over an arbitrary commutative ring k. However in this general case many of the γ_{dei} may be 0.

In the general case η_i is the composition

$$\begin{array}{ccccccc} R_{d+e-2i} & \xrightarrow{\varphi_i} & S_{de} & \xrightarrow{\Omega^i} & S_{d-i,e-i} & \xrightarrow{\sigma} & R_{d+e-2i} \\ & f \mapsto \Delta^i D^{e-i} f & & & & & \end{array}$$

Since $D^{e-i}f \in S_{d-i,e-i}$ the first application of Ω yields (by Lemma 2)

$$\Omega(\Delta^i D^{e-i} f) = i\,(d+e-i+1)\,\Delta^{i-1} D^{e-i} f + \Delta^i\,\Omega(D^{e-i} f)$$

Applying Ω iteratively i times, the second term on the righthand side becomes confusing. Fortunately with don't need to bother with it due to the following lemma:

Lemma 3 *Let $f \in R_{d+e-2i}$. Then for each $j = 0, \ldots, i$ there is an $F_j \in S_{d-i-j,e-i-j}$ such that*

$$\Omega^j(\Delta^i D^{e-i} f) = \frac{i!}{(i-j)!}\,\frac{(d+e-i+1)!}{(d+e-i+1-j)!}\,\Delta^{i-j} D^{e-i} f + \Delta^{i-j+1} F_j.$$

Proof. For $j = 0$ the assertion holds with $F_0 = 0$.

Now assume that $j \ge 1$, and by induction that the assertion is proved for $j-1$ instead of j. That is

$$\Omega^{j-1}(\Delta^i D^{e-i} f) = \frac{i!}{(i-j+1)!}\,\frac{(d+e-i+1)!}{(d+e-i+2-j)!}\,\Delta^{i-j+1} D^{e-i} f + \Delta^{i-j+2} F_{j-1}.$$

Applying Ω to this equation and using (iii) of Lemma 2 we get

$$\begin{aligned} \Omega^j(\Delta^i D^{e-i} f) = {} & \frac{i!}{(i-j+1)!}\,\frac{(d+e-i+1)!}{(d+e-i+2-j)!}\,\Omega(\Delta^{i-j+1} D^{e-i} f) \\ & + \Omega(\Delta^{i-j+2} F_{j-1}) \end{aligned}$$

$$\begin{aligned} = {} & \frac{i!}{(i-j+1)!}\,\frac{(d+e-i+1)!}{(d+e-i+2-j)!} \\ & \times \left[(i-j+1)(d-i+e-i+i-j+2)\Delta^{i-j} D^{e-i} f + \Delta^{i-j+1}\Omega(D^{e-i} f)\right] \\ & + \left[(i-j+2)(d-i+e-i+i-j+1)\Delta^{i-j+1} F_{j-1} + \Delta^{i-j+2}\Omega(F_{j-1})\right] \end{aligned}$$

The first (of four) summands yields

$$\frac{i!}{(i-j)!}\,\frac{(d+e-i+1)!}{(d+e-i+1-j)!}\,\Delta^{i-j}D^{e-i}f.$$

The remaining three summands, up to integer multiples, are

$$\Delta^{i-j+1}\Omega(D^{e-i}f),\quad \Delta^{i-j+1}F_{j-1},\quad \Delta^{i-j+1}\Delta\,\Omega(F_{j-1}),$$

and

$$\Omega(D^{e-i}f),\quad F_{j-1},\quad \Delta\,\Omega(F_{j-1})$$

are in $S_{d-i-1,e-i-1}$. $\diamond$

For the *proof of the theorem* we apply the lemma with $j=i$ and get

$$\Omega^i(\Delta^i D^{e-i}f) = i!\,\frac{(d+e-i+1)!}{(d+e-2i+1)!}\,D^{e-i}f + \Delta F_i.$$

Using $\sigma(\Delta)=0$ and Corollary 1 of Proposition 2 we finally get

$$\begin{aligned}\eta_i(f) = \sigma\circ\Omega^i(\Delta^i D^{e-i}f) &= i!\,\frac{(d+e-i+1)!}{(d+e-2i+1)!}\,\frac{(d+e-2i)!}{(d-i)!}\,f+0\\ &= \frac{i!}{(d-i)!}\,\frac{(d+e-i+1)!}{d+e-2i+1}\,f,\end{aligned}$$

and the proof of the theorem is complete. $\diamond$

Examples The formula in the theorem reproduces the values γ_{de0} and γ_{de1} from above. As another example take

$$\gamma_{dee} = \frac{e!}{(d-e)!}\,\frac{(d+1)!}{d-e+1} = \frac{e!\,(d+1)!}{(d-e+1)!}$$

We might also look at the compositions

$$R_{d+e-2i} \xrightarrow{\varphi_i} S_{de} \xrightarrow{\Omega^j} S_{d-j,e-j} \xrightarrow{\sigma} R_{d+e-2j}$$

for $j\neq i$. In the case $j<i$ we use Lemma 3 (and abbreviate the integer coefficient by c):

$$\Omega^j(\varphi_i(f)) = c\,\Delta^{i-j}D^{e-i}f + \Delta^{i-j+1}F_j = \Delta\,(\ldots) \overset{\sigma}{\mapsto} 0.$$

Furthermore since f is independent from the indeterminates Y we have

$$\Omega(\Omega^i(\varphi_i(f))) = \Omega(c'\,f) = 0,$$

hence $\Omega^j(\varphi_i(f))=0$ for $j>i$. This proves:

Corollary 4 *For $d, e, i \in \mathbb{N}$, $0 \le i \le e \le d$, and $j \in \mathbb{N}$, $0 \le j \le e$, $j \ne i$,*

$$\sigma \circ \Omega^j \circ \varphi_i = 0 \quad \text{on } R_{d+e-2i}.$$

Now, if k is a field of characteristic 0, by Corollary 2 (or Theorem 1) each $F \in S_{de}$ has a unique decomposition as

$$F = \sum_{i=0}^{e} \varphi_i(f_i) \quad \text{with } f_i \in R_{d+e-2i}.$$

Theorem 2 and Corollary 4 allow to express the f_i in terms of F: For $j = 0, \ldots, e$ we conclude that

$$\sigma \circ \Omega^j(F) = \sum_{i=0}^{e} \sigma \circ \Omega^j \circ \varphi_i(f_i) = \eta_j(f_j) = \gamma_{dej} f_j.$$

Hence $f_j = \sigma \circ \Omega^j(F)/\gamma_{dej}$, if $\gamma_{dej} \in k^\times$, thus

$$F = \sum_{i=0}^{e} \frac{1}{\gamma_{dei}} \, \varphi_i \circ \sigma \circ \Omega^i(F).$$

We have proved

Corollary 5 (Cayley-Sylvester) *Let k be a field of characteristic 0. Then each $F \in S_{de}$ decomposes as*

$$F = \sum_{i=0}^{e} \frac{1}{\gamma_{dei}} \, \psi_i(F)$$

where $\psi_i(F) = \Delta^i D^{e-i}(\sigma \circ \Omega^i(F)) \in \varphi_i(R_{d+e-2i})$.

Examples Let us look at the decomposition of Corollary 5 for some simple special cases.

- For $F \in S_{11}$ we have the coefficients $\gamma_{110} = 2$ and $\gamma_{111} = 2$, hence

$$F = \frac{1}{2} \psi_0(F) + \frac{1}{2} \psi_1(F).$$

- For $F \in S_{21}$ we have the coefficients $\gamma_{210} = 3$ and $\gamma_{211} = 3$, hence

$$F = \frac{1}{3} \psi_0(F) + \frac{1}{3} \psi_1(F).$$

- For $F \in S_{22}$ we have the coefficients $\gamma_{220} = 12$, $\gamma_{221} = 8$, and $\gamma_{222} = 12$, hence

$$F = \frac{1}{12} \psi_0(F) + \frac{1}{8} \psi_1(F) + \frac{1}{12} \psi_2(F).$$

- For $F \in S_{32}$ we have the coefficients $\gamma_{320} = 20$, $\gamma_{321} = 15$, and $\gamma_{322} = 24$, hence

$$F = \frac{1}{20} \psi_0(F) + \frac{1}{15} \psi_1(F) + \frac{1}{24} \psi_2(F).$$

4 Transvection

We consider the map

$$\tilde{\mu}\colon R_d \times R_e \longrightarrow S_{de}, \quad \tilde{\mu}(f,h) = f(X)\,h(Y) = f\tilde{h}.$$

(That is, we multiply f with h after replacing the indeterminates X_1, X_2 by Y_1, Y_2 in h, yielding $\tilde{h} = h(Y)$.) In characteristic 0 and for $1 \le e \le d$ Corollary 5 gives a unique decomposition of this product as

$$\tilde{\mu}(f,h) = \sum_{i=0}^{e} \frac{1}{\gamma_{dei}}\,\varphi_i \circ \tau_i(f,h) \quad \text{with } \tau_i(f,h) = \sigma \circ \Omega^i \circ \tilde{\mu}(f,h) \in R_{d+e-2i}.$$

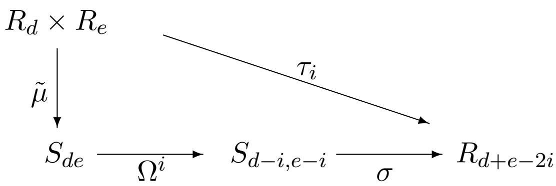
This definition of the τ_i also makes sense for $d < e$ and in any characteristic:

Definition For all $d, e \ge 0$ the map

$$\tau_i\colon R_d \times R_e \longrightarrow R_{d+e-2i}, \quad \tau_i(f,h) = \sigma \circ \Omega^i \circ \tilde{\mu}(f,h)$$

is called the i^{th} **transvection**, its images i^{th} **transvectants**.

If $2i > d+e$, then $\tau_i = 0$. The following commutative diagram illustrates the definition of the maps τ_i:



Clearly the maps τ_i are bilinear and relatively G-equivariant of weight $-i$ because $\tilde{\mu}$ is bilinear and equivariant, Ω^i is linear and relatively equivariant of weight $-i$, and σ is linear and equivariant.

The most elementary special case is

$$\tau_0(f,h) = \sigma \circ \tilde{\mu}(f,h) = \sigma(f\tilde{h}) = fh \in R_{d+e}$$

so the 0^{th} transvectant of two binary forms is simply their product. For $i = 1$ we use Lemma 2 and get

$$\Omega(f\tilde{h}) = \partial_1 f\,\tilde{\partial}_2\tilde{h} - \partial_2 f\,\tilde{\partial}_1\tilde{h} = \begin{vmatrix} \partial_1 f & \tilde{\partial}_1\tilde{h} \\ \partial_2 f & \tilde{\partial}_2\tilde{h} \end{vmatrix},$$

$$\tau_1(f,h) = \sigma(\Omega(f\tilde{h})) = \partial_1 f\,\partial_2 h - \partial_2 f\,\partial_1 h = \begin{vmatrix} \partial_1 f & \partial_1 h \\ \partial_2 f & \partial_2 h \end{vmatrix} \in R_{d+e-2}.$$

For the 2^{nd} transvectant we compute

$$\begin{aligned} \Omega^2(f\tilde{h}) &= \Omega(\partial_1 f\,\tilde{\partial}_2\tilde{h}) - \Omega(\partial_2 f\,\tilde{\partial}_1\tilde{h}) \\ &= \partial_1^2 f\,\tilde{\partial}_2{}^2\tilde{h} - \partial_2\partial_1 f\,\tilde{\partial}_1\tilde{\partial}_2\tilde{h} - \partial_1\partial_2 f\,\tilde{\partial}_2\tilde{\partial}_1\tilde{h} + \partial_2^2 f\,\tilde{\partial}_1{}^2\tilde{h}, \end{aligned}$$

$$\tau_2(f,h) = \partial_1^2 f\,\partial_2^2 h - 2\,\partial_1\partial_2 f\,\partial_1\partial_2 h + \partial_2^2 f\,\partial_1^2 h \in R_{d+e-4}.$$

Proposition 6 *The* 0^{th}, 1^{st}, *and* 2^{nd} *transvectants of two binary forms* $f \in R_d$ *and* $g \in R_e$ *are*

(i) $\tau_0(f,h) = fh \in R_{d+e}$,

(ii) $\tau_1(f,h) = \partial_1 f\, \partial_2 h - \partial_2 f\, \partial_1 h \in R_{d+e-2}$, *the Jacobian of the pair* (f,h),

(iii) $\tau_2(f,h) = \partial_1^2 f\, \partial_2^2 h - 2\,\partial_1\partial_2 f\, \partial_1\partial_2 h + \partial_2^2 f\, \partial_1^2 h \in R_{d+e-4}$.

Corollary 6 *The* 0^{th}, 1^{st}, *and* 2^{nd} *transvectants of a binary form* $f \in R_d$ *with itself are*

(i) $\tau_0(f,f) = f^2 \in R_{2d}$,

(ii) $\tau_1(f,f) = 0 \in R_{2d-2}$,

(iii) $\tau_2(f,f) = 2\,[\partial_1^2 f\, \partial_2^2 f - (\partial_1\partial_2 f)^2] \in R_{2d-4}$ *(twice the Hessian).*

Examples For $f = a_0X_1^2 + a_1X_1X_2 + a_2X_2^2$, $h = b_0X_1^2 + b_1X_1X_2 + b_2X_2^2 \in R_2$ we get

- $\tau_0(f,h) = a_0b_0X_1^4 + (a_0b_1 + a_1b_0)X_1^3X_2 + (a_0b_2 + a_1b_1 + a_2b_0)X_1^2X_2^2$ $+ (a_1b_2 + a_2b_1)X_1X_2^3 + a_2b_2X_2^4 \in R_4$,
- $\tau_1(f,h) =$ $(2a_0X_1 + a_1X_2)(b_1X_1 + 2b_2X_2) - (a_1X_1 + 2a_2X_2)(2b_0X_1 + b_1X_2)$ $= 2\,(a_0b_1 - a_1b_0)X_1^2 + 4\,(a_0b_2 - a_2b_0)X_1X_2 + 2\,(a_1b_2 - a_2b_1)X_2^2$ $\in R_2$,
- $\tau_2(f,h) = 4\,(a_0b_2 + a_2b_0) - 2\,a_1b_1 \in R_0 = k$,
- $\tau_2(f,f) = 8\,a_0a_2 - 2\,a_1^2 \in R_0 = k$.

To explore the symmetry properties of the transvections τ_i we consider the involution $\varepsilon : X_i \longleftrightarrow Y_i$ of the k-algebra $S = k[X,Y]$, that is $\varepsilon(F(X,Y)) = F(Y,X)$.

Lemma 4 *For* ε *we have*

(i) $\Omega^i \circ \varepsilon = (-1)^i\, \varepsilon \circ \Omega^i$.

(ii) *The restriction of* $\sigma \circ \varepsilon$ *to the subalgebra* $R = k[X]$ *is the identity map.*

(iii) $\sigma \circ \varepsilon = \sigma \circ \varepsilon \circ \sigma$ *on* S.

Proof. (i) It suffices to prove the assertion for $i = 1$. For this we consider the monomial $F = X_1^{\alpha_1}X_2^{\alpha_2}Y_1^{\beta_1}Y_2^{\beta_2}$. Then

$$\begin{aligned} F &\overset{\Omega}{\mapsto} \alpha_1\beta_2 X_1^{\alpha_1-1}X_2^{\alpha_2}Y_1^{\beta_1}Y_2^{\beta_2-1} - \beta_1\alpha_2 X_1^{\alpha_1}X_2^{\alpha_2-1}Y_1^{\beta_1-1}Y_2^{\beta_2} \\ &\overset{\varepsilon}{\mapsto} \alpha_1\beta_2 Y_1^{\alpha_1-1}Y_2^{\alpha_2}X_1^{\beta_1}X_2^{\beta_2-1} - \beta_1\alpha_2 Y_1^{\alpha_1}Y_2^{\alpha_2-1}X_1^{\beta_1-1}X_2^{\beta_2}, \\ F &\overset{\varepsilon}{\mapsto} Y_1^{\alpha_1}Y_2^{\alpha_2}X_1^{\beta_1}X_2^{\beta_2} \\ &\overset{\Omega}{\mapsto} \beta_1\alpha_2 Y_1^{\alpha_1}Y_2^{\alpha_2-1}X_1^{\beta_1-1}X_2^{\beta_2} - \beta_2\alpha_1 Y_1^{\alpha_1-1}Y_2^{\alpha_2}X_1^{\beta_1}X_2^{\beta_2-1}. \end{aligned}$$

Hence $\Omega \circ \varepsilon = -\varepsilon \circ \Omega$.

(ii) For $f \in R$ we conclude $\sigma(\varepsilon(f)) = \sigma(f(Y)) = f(X) = f$.

(iii) Since σ and ε are k-algebra homomorphisms it suffices to prove the assertion for the generators X_i and Y_i.

For X_i we have $\sigma(X_i) = X_i$, and by (ii) both sides evaluate to X_i.

For Y_i we have $\sigma(Y_i) = X_i$, thus again both sides of the equation evaluate to X_i. $\diamond$

Proposition 7 *For $f \in R_d$, $h \in R_e$,*

$$\tau_i(h, f) = (-1)^i \, \tau_i(f, h).$$

Proof. Using Lemma 4 we get

$$\begin{aligned}\tau_i(h, f) &= \sigma \circ \Omega^i(h\tilde{f}) = \sigma \circ \Omega^i \circ \varepsilon(f\tilde{h}) = (-1)^i \, \sigma \circ \varepsilon \circ \Omega^i(f\tilde{h}) \\ &= (-1)^i \, \sigma \circ \varepsilon \circ \sigma \circ \Omega^i(f\tilde{h}) = (-1)^i \, \sigma \circ \varepsilon(\tau_i(f, h)) \\ &= (-1)^i \, (\tau_i(f, h))\end{aligned}$$

since $\tau_i(f, h) \in R$. $\diamond$

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