Fixed Binary Forms

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The space R_d of homogenous polynomials in two variables X and Y, or of binary forms, of degree d over an algebraically closed field k is an irreducible SL_2-module with a few exceptions in prime characteristics. Therefore the question of trivial submodules, or in other words of non-zero fixed points for SL_2, makes sense.

1 The Operation of the Group SL_2 and its Lie Algebra $\mathfrak{sl}_2$

Let k be an algebraically closed field. We consider the group $G = SL_2(k)$ of 2×2-matrices with determinant 1 over k. The matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

acts on the 2-dimensional vector space k^2 by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Denote the coordinate functions $k^2 \longrightarrow k$ by X and Y, where

$$X \begin{pmatrix} x \\ y \end{pmatrix} = x, \quad Y \begin{pmatrix} x \\ y \end{pmatrix} = y$$

for all $x, y \in k$. Since the inverse of g is

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

the induced ("contragredient") action on the space of linear forms spanned by the coordinate functions X and Y is given by

$$X \mapsto dX - bY,$$
$$Y \mapsto -cX + aY.$$

(In general a function $f\colon k^2 \longrightarrow k$ is transformed to $f \circ g^{-1}$.) This action extends to the polynomial ring $k[X,Y]$ as group of automorphisms. In particular for the powers of the coordinate functions we get the formulas

$$\begin{aligned} X^r \mapsto (dX - bY)^r \quad &= d^r X^r - r d^{r-1} b X^{r-1} Y + \cdots + (-1)^r b^r Y^r \\ &= \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} b^\nu d^{r-\nu} X^{r-\nu} Y^\nu, \\ Y^s \mapsto (-cX + aY)^s \quad &= (-c)^s X^s + s(-c)^{s-1} a X^{s-1} Y + \cdots + a^s Y^s \\ &= \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} a^\nu c^{s-\nu} X^{s-\nu} Y^\nu. \end{aligned}$$

Thus depending on the prime divisors of the binomial coefficients there are some anomalies in prime characteristics.

The Lie algebra $\mathfrak{sl}_2(k)$ consists of the 2×2-matrices with trace 0,

$$\mathfrak{sl}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \,\middle|\, a, b, c \in k \right\}.$$

It acts on the polynomial ring $k[X,Y]$ by derivations, starting with the formulas

$$\begin{aligned} X &\mapsto -aX - bY, \\ Y &\mapsto -cX + aY, \end{aligned} \quad \text{for } A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(k).$$

(The easiest way to remember these formulas is by using dual numbers [1, Section 9.5], that is considering $SL_2(k[\delta])$ where $\delta^2 = 0$.) In particular

$$\begin{aligned} X^r &\mapsto r X^{r-1}(-aX - bY), \\ Y^s &\mapsto s\, Y^{s-1}(-cX + aY). \end{aligned}$$

Let $R = k[X,Y]$ be the polynomial ring and R_d be its homogeneous part of degree d, an SL_2-invariant subspace of R with $\dim_k R_d = d + 1$.

2 "Fixed Points" for the Lie Algebra

The Lie algebra $\mathfrak{sl}_2(k)$ is spanned (as a vector space over k) by the three matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

whose effects on the variables X and Y are given by the formulas

$$HX = -X, \quad EX = -Y, \quad FX = 0,$$

$$HY = Y, \quad EY = 0, \quad FY = -X.$$

By definition an element A of the Lie algebra "stabilizes" a vector v in an SL_2-module if and only if $Av = 0$, in other words, if the Lie algebra annihilates v. (Think of Av as a displacement.) Fixed points of the group G are annihilated by the Lie algebra $\mathfrak{g}$, expressed as a formula:

$$G_v = G \Longrightarrow \mathfrak{g}_v = \mathfrak{g},$$

or more generally,

$$\mathrm{Lie}(G_v) \subseteq \mathfrak{g}_v$$

In prime characteristics the converse is not always true.

Now v is annihilated by $\mathfrak{g}$ if and only if it is annihilated by the three matrices H, E, and F. The elements $v \in R_d$ have the form

$$v = \sum_{\nu=0}^{d} a_\nu X^{d-\nu} Y^\nu.$$

The matrices H, E, and F act on R as derivations. Thus for their effects on the basis elements we have the formulas:

$$HX^r = -rX^r, \quad EX^r = -rX^{r-1}Y, \quad FX^r = 0,$$

$$HY^s = sY^s, \quad EY^s = 0, \quad FY^s = -sXY^{s-1},$$

$$H(X^rY^s) = (s-r)X^rY^s, \quad E(X^rY^s) = -rX^{r-1}Y^{s+1}, \quad F(X^rY^s) = -sX^{r+1}Y^{s-1}.$$

The effects on the typical element v are then given by

$$\begin{aligned} H(\sum_{\nu=0}^{d} a_\nu X^{d-\nu}Y^\nu) &= \sum_{\nu=0}^{d} (2\nu - d)\, a_\nu X^{d-\nu}Y^\nu \\ E(\sum_{\nu=0}^{d} a_\nu X^{d-\nu}Y^\nu) &= \sum_{\nu=0}^{d-1} (\nu - d)\, a_\nu X^{d-\nu-1}Y^{\nu+1} \\ F(\sum_{\nu=0}^{d} a_\nu X^{d-\nu}Y^\nu) &= \sum_{\nu=1}^{d} (-\nu)\, a_\nu X^{d-\nu+1}Y^{\nu-1} \end{aligned}$$

This yields the equivalences

$$\begin{aligned} Hv = 0 &\Longleftrightarrow (2\nu - d)\cdot a_\nu = 0 \quad \text{for } \nu = 0, \ldots, d, \\ Ev = 0 &\Longleftrightarrow (\nu - d)\cdot a_\nu = 0 \quad \text{for } \nu = 0, \ldots, d-1, \\ Fv = 0 &\Longleftrightarrow \nu \cdot a_\nu = 0 \qquad\quad \text{for } \nu = 1, \ldots, d. \end{aligned}$$

Proposition 1 *The binary form* $\sum_{\nu=0}^{d} a_\nu X^{d-\nu} Y^\nu$ *is annihilated by* $\mathfrak{sl}_2(k)$ *if and only if*

$$da_0 = 0, \quad da_d = 0, \quad \nu a_\nu = da_\nu = 0 \quad \text{for } \nu = 1, \ldots, d-1.$$

We denote by $V^{\mathfrak{g}}$ the subspace of points of a G-Module V annihilated by $\mathfrak{g}$. We have shown:

Corollary 1 *If* $\operatorname{char} k \nmid d$*, then* $R_d^{\mathfrak{g}} = 0$.

Corollary 2 *If* $p = \operatorname{char} k \mid d$*, then* $v \in R_d^{\mathfrak{g}}$ *if and only if*

$$\nu a_\nu = 0 \quad \text{for } \nu = 1, \ldots, d-1.$$

Thus $R_d^{\mathfrak{g}}$ *is spanned by the monomials* $X^{d-\nu}Y^\nu$ *with* $p \mid \nu$*. In particular* $X^d, Y^d \in R_d^{\mathfrak{g}}$.

Examples

$d = 2:$ $R_2^{\mathfrak{g}} = 0$ except when $p = 2$. In this exceptional case $R_2^{\mathfrak{g}}$ is spanned by X^2 and Y^2.

$d = p$ where $p = \operatorname{char} k$: The same consideration shows that $R_p^{\mathfrak{g}}$ is spanned by X^p and Y^p.

$d = 4:$ $R_4^{\mathfrak{g}} = 0$ except when $p = 2$. In this case $R_4^{\mathfrak{g}}$ is spanned by X^4, X^2Y^2, and Y^4.

$d = 6:$ $R_6^{\mathfrak{g}} = 0$ except when $p = 2$ or 3.

For $p = 2$ the space $R_6^{\mathfrak{g}}$ is spanned by X^6, X^4Y^2, X^2Y^4, and Y^6.

For $p = 3$ the space $R_6^{\mathfrak{g}}$ is spanned by X^6, X^3Y^3, and Y^6.

3 Fixed Points for the Group

We denote by V^G the subspace of fixed points of G in a G-module V. From Corollaries 1 and 2 we immediately get:

Corollary 3 *If* $\operatorname{char} k \nmid d$*, then* $R_d^G = 0$.

Corollary 4 *If* $p = \operatorname{char} k \mid d$*, then* R_d^G *is contained in the subspace* $V = R_d^{\mathfrak{g}}$ *spanned by the monomials* $X^{d-\nu}Y^\nu$ *with* $p \mid \nu$*. In particular* $R_d^G = V^G$.

Now consider the action of the maximal torus $T \leq G$ consisting of the matrices

$$\Delta(t) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \quad \text{for } t \in k^\times.$$

that map $X^r \mapsto t^{-r}X^r$ and $Y^r \mapsto t^rY^r$. Thus applying $\Delta(t)$ to

$$v = \sum_{\nu=0}^{d} a_\nu X^{d-\nu}Y^\nu$$

yields the vector

$$\sum_{\nu=0}^{d} a_\nu t^{2\nu-d} X^{d-\nu} Y^\nu.$$

Hence v is fixed by T if and only if

$$a_\nu t^{2\nu-d} = a_\nu \quad \text{for all } t \in k^\times.$$

Since k is assumed as algebraically closed, hence infinite, this forces $a_\nu = 0$ except when $t^{2\nu-d} = 1$ constant, i. e. when $d = 2\nu$.

Proposition 2 *The binary forms of degree d that are fixed by T form the subspace*

$$R_d^T = \begin{cases} 0 & \text{if } d \text{ is odd,} \\ kX^rY^r & \text{if } d = 2r \text{ is even.} \end{cases}$$

Since $R_d^G \subseteq R_d^T$ we have only to study the form X^rY^r (or its scalar multiples). We expose it to the maximal unipotent subgroup consisting of the matrices

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{for } b \in k,$$

getting

$$(X - bY)^r\, Y^r = (X^r \pm \cdots \pm b^rY^r)\, Y^r = X^rY^r \pm \cdots \pm b^rY^{2r}.$$

This is X^rY^r if and only if $b = 0$. Hence X^rY^r is not fixed by G.

Theorem 1 $R_d^G = 0$.

References

[1] J. E. Humphreys: *Linear Algebraic Groups*. Springer-Verlag, New York 1975.