Richardson's Finiteness Theorem

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Richardson's Finiteness Theorem says (among other things) that the number of conjugacy classes of nilpotent elements in the Lie algebra $\mathfrak{g}$ of a semisimple algebraic group G is finite, provided that the base field k is algebraically closed of good characteristic, see Theorem 1.

Here we derive the Finiteness Theorem from a slightly more general statement, see Proposition 1, that requires no assumption on the characteristic. However in applying this to nilpotent classes we need Proposition 2 where (in prime characteristic) we cannot remove the dependence on the classification of semisimple groups and on the characteristic being good.

Note that in characteristic 0 Lemma 1 and Proposition 2 are obsolete—as a non-degenerate bilinear form we may take the Killing form—and Proposition 1 can be somewhat simplified; so in characteristic 0 Theorem 1 has a simple proof independent from the classification.

For other applications of Proposition 1 see [3] and [5].

Recall that for the action of an affine algebraic group G on a rational G-module the orbit map $G \longrightarrow G \cdot x$ for $x \in V$ is separable, if $\mathfrak{g}_x = \operatorname{Lie} G_x$ or, equivalently, $\dim G \cdot x = \dim \mathfrak{g} \cdot x$ [1, p. 180]. (In the present context we may take this as a definition.) The characteristic p is good for G, if p doesn't occur as coefficient of the expansion of the highest root in terms of a basis of the root system. This excludes the following primes, if G has a component of type

- $\mathbf{A}_l$: none,
- $\mathbf{B}_l$, $\mathbf{C}_l$, $\mathbf{D}_l$: 2,
- $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$ $\mathbf{E}_7$: 2, 3,
- $\mathbf{E}_8$: 2, 3, 5.

Lemma 1 *For the adjoint representation of $\mathbf{GL}_n$ all orbit maps are separable.*

Proof. Identify the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ with the space of all $n \times n$ matrices over k and the adjoint action of $g \in \mathbf{GL}_n$ with the conjugation $x \mapsto gxg^{-1}$. Then the stabilizer of x is $\{g \in \mathbf{GL}_n \mid gx = xg\}$, and is open in the vector subspace $\mathfrak{g}_x = \{z \in \mathfrak{g} \mid [zx] = 0\}$, because it is cut out by the nonvanishing of the determinant. $\diamond$

Proposition 1 *Let H be an affine algebraic group, and V be a rational H-module. Let G be a closed subgroup of H, and W be a G-submodule of V. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. Assume that for some $x \in W$*

(i) *the orbit map $H \longrightarrow H \cdot x$ is separable,*

(ii) *$\mathfrak{h} \cdot y \cap W \subseteq \mathfrak{g} \cdot y$ for each $y \in W \cap H \cdot x$.*

Then $W \cap H \cdot x$ consists of finitely many G-orbits, and the orbit map $G \longrightarrow G \cdot x$ is separable.

Proof. The separability of the orbit map $H \longrightarrow H \cdot x$ doesn't change when we replace x with any other point of the orbit $H \cdot x$.

Now let X be the connected component of x in $G \cdot x$, and Z be an irreducible component of $W \cap H \cdot x$ that contains x. We have *to show that* $\dim \mathfrak{g} \cdot x = \dim G \cdot x = \dim Z$. From this follows that X is open in Z, and because x may be replaced with any other point of Z, even $X = Z$. Therefore the number of G-orbits in $W \cap H \cdot x$ equals at most the (finite) number of irreducible components.

Now we may identify the tangent space $T_0(H \cdot x - x)$ with the subspace $\mathfrak{h} \cdot x$ of V in a canonical way. Therefore

$$T_x(Z) \cong T_0(Z - x) \subseteq W \cap \mathfrak{h} \cdot x \subseteq \mathfrak{g} \cdot x,$$

$$\dim Z \leq \dim T_x(Z) \leq \dim \mathfrak{g} \cdot x = \dim X \leq \dim Z.$$

Therefore everywhere in the last row we have equality. $\diamond$

The following result depends on the classification of semisimple algebraic groups.

Proposition 2 *Let G be an almost simple algebraic group in good characteristic, and G not of type $\mathbf{A}_l$, in particular* $\operatorname{char} k \neq 2$. *Then there is a rational G-Module such that the trace form $\tau(x, y) := \operatorname{Tr}(x \circ y)$ is a nondegenerate bilinear form on the Lie algebra $\mathfrak{g}$ of G.*

Proof. For the groups of types $\mathbf{B}_l$, $\mathbf{C}_l$, and $\mathbf{D}_l$—that is for $\mathbf{SO_n}$ and $\mathbf{Sp_n}$—we may take the natural representation. For the exceptional types the adjoint representation does the job, see [4] or [6]. $\diamond$

Theorem 1 (Richardson) *Let G be a semisimple algebraic group in good characteristic with Lie algebra $\mathfrak{g}$. Then the number of G-orbits of nilpotent elements in $\mathfrak{g}$ is finite.*

Proof. Without restriction we may assume G almost simple. Because the case of $\mathbf{A}_l$ is elementary linear algebra we may also exclude this case. Then Proposition 2 gives a rational G-module V such that the trace form on $\mathfrak{g}$ is nondegenerate. The trace form is also nondegenerate on $\mathfrak{gl}(V)$. Therefore we have $\mathfrak{gl}(V) = \mathfrak{g} \oplus M$ as G-modules with

$M = \mathfrak{g}^\perp$. The assertion follows from Proposition 1 with $H = GL(V)$: Condition (i) is fulfilled by Lemma 1; for condition (ii) we have $\mathfrak{gl}(V) \cdot y = \mathfrak{g} \cdot y + M \cdot y$, hence $\mathfrak{gl}(V) \cdot y \cap \mathfrak{g} \subseteq \mathfrak{g} \cdot y$. $\diamond$

The restriction on the characteristic is unnecessary; however this needs a tedious case-by-case calculation that was completed by Holt and Spaltenstein in [2].

References

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