Analytical Theory of transverse Anderson localization of light

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We develop an analytical theory for describing the transverse localization properties of light beams in optical fibers with lateral disorder. This theory, which starts from the widely used paraxial approximation for the Helmholtz equation of the electric field, is a combination of an effective-medium theory for transverse disorder with the self-consistent localization theory of Vollhardt and Wölfle. We obtain explicit expressions for the dependence of the transverse localization length on the direction along the fiber. These results are in agreement with simulational data published recently by Karbasi et al. In particular we explain the focussing mechanism leading to the establishment of narrow transparent channels along the sample.

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By exploiting the paraxial approximation it is possible to treat electromagnetic classical waves in a quasi one-dimensional environment by a Schrödinger-like equation [1], in which the longitudinal coordinate plays the role of a time coordinate. This makes it quite easy to simulate and predict the behavior of light beams in laser cavities, in optical fibers, or in many other technological relevant systems with transverse confinement on a micron scale.

Presently the paraxial equation is exploited in a variety of propagation invariant systems such as photonic crystal fibers [2] or step index fibers [3], which are nowadays the backbone of our communication network. Usually optical fibers are homogeneous strands of a transparent material, not only because unwanted scattering disturbs the optical transmission, but also because symmetric systems are easier to understand and study. However, light guiding may be achieved also for transversely inhomogeneous or even disordered structures, exploiting, in fact, Anderson localization.

In the last 50 years Anderson localization, i.e. the disorder-induced arrest of wave diffusion [4] has grown to be a fascinating field of the physics of electrons [5–9], atomic matter waves [10, 11] and classical waves [12–20]. It was proposed by Abdullaev and Abdullaev [21] and later, independently, by DeRaedt et al. [22] that in an optical fiber, which is constructed in such a way that the dielectric constant varies in the transverse direction, but is invariant along the fiber axis, the electromagnetic field should exhibit Anderson localization in the x, y direction, but remain delocalized in the z direction. This is of interest for applications, because by the localization mechanism a light beam stays confined in the transverse direction as in a waveguide. It was not until eight years later that Schwartz et al. [23] found a realization in a disordered optical lattice. In these experiments the disorder was rather weak, which led to localization radii of a fraction of a millimeter. Later Karbasi et al. [24] devised samples of different polymer constituents to achieve localization lengths of the order of a visible light wavelength. Karbasi et al. [25] showed that by means of such fibers image transport for endoscopy is as good as in commercially observable multicore fibers and that the quality can be optimized by increasing the disorder contrast e.g. by using tiny glass fibers with air in between.

While the impact of the disorder and the sample geometry on the localization length and the ray characteristics in the transverse-localized samples was studied by numerical simulation by DeRaedt et al. [22] and Karbasi et al. [26], a detailed analytic theory is missing, although an adequate analytical theory for quantum [9, 27–31] and classical waves [9, 32] exists in principle and has been successfully applied to localization dynamics in geometries different to those relevant for transverse localization [33–35]. Such a theory is of importance, because explicit formulae for the impact of the external parameters onto the formation of the localized channels are indispensable for the design of a disorder-based optical waveguide. This is relevant for the optimization of Anderson-localization based image-transport devices, the study of the channel-to-channel crosstalk and, more generally, for a deeper understanding of the transverse localization phenomenon.

In this letter, starting from the equation that has been simulated in Refs. [22, 26] we formulate such an analytic theory of transverse Anderson localization by combining the effective-medium theory of waves in a random environment [12, 13, 36–38] with the localization theory of Vollhardt and Wölfle [27]. This will be achieved in three steps: Firstly we derive a theory for the averaged single-particle Green's function of the stochastic paraxial wave equation using the self-consistent Born approximation (SCBA) [12, 13, 36–38] within a field-theoretical framework. We then derive an expression for the unrenormalized transverse diffusivity by going beyond the SCBA saddle point. Finally we use the localization theory of Vollhardt and Wölfle [27] for the two-particle Green's function with the unrenormalized diffusivity as input.

We consider a transparent material, which has a spatially varying index of refraction n in the transverse direction denoted by the vector $\boldsymbol{\rho} = (x, y)$. The components of the electric field $E_\alpha(\boldsymbol{\rho}, k_z) = \int dz e^{ik_z z} E_\alpha(\boldsymbol{\rho}, z)$ are assumed to obey the following stochastic Helmholtz equation

$$\left(E + k_0^2 \Delta(\boldsymbol{\rho}) + \nabla_{\boldsymbol{\rho}}^2\right) E_\alpha(\boldsymbol{\rho}, k_z) = 0\,. \tag{1}$$

with the spectral (mode) parameter $E = k_0^2 - k_z^2 = k_\perp^2$, where $k_0 = \omega\langle n\rangle/c_0$ is the average wavenumber inside the medium, $\langle n\rangle$ is the average index of refraction and c_0 is the light velocity in vacuum. $\Delta(\boldsymbol{\rho}) = [n^2(\boldsymbol{\rho}) - \langle n\rangle^2]/\langle n\rangle^2$ is the relative variation of the permittivity. $k_z = k_0\cos(\theta)$ is the z component of the wave vector and θ is the initial azimuthal angle, in terms of which we have $E = k_0^2 \sin(\theta)^2$. For small θ (paraxial limit) the sine can be replaced by its argument, and we have $E \approx -2k_0\Delta k_z$, where the wavenumber $\Delta k_z = k_z - k_0$ refers to the Fourier component of the *envelope* $A(\boldsymbol{\rho}, z) =$

$E_\alpha(\boldsymbol{\rho}, z)e^{-ik_0 z}$. The function $A(\boldsymbol{\rho}, z)$ describes the *beats* of $E_\alpha(\boldsymbol{\rho}, z)$ in z direction. In the small θ limit it obeys the paraxial (Schrödinger) equation

$$\left(i\frac{\partial}{\partial\tau} + k_0^2\Delta n(\boldsymbol{\rho}) + \nabla_\rho^2\right)A(\boldsymbol{\rho}, z) = 0\,, \qquad (2)$$

which has been used by DeRaedt et al. [22] and Karbasi et al. [24, 26] in their simulation of transverse localization. The quantity $\tau = z/2k_0$ (which has the dimension of a squared length) plays the role of a "time". Equation (2) describes the spread of the amplitude $A(\boldsymbol{\rho}, z)$ in the presence of a random potential as the time τ goes by, i.e. as a function of z. It is worth noting that this equation is identical to the Schrödinger equation for a two-dimensional electron system in a random potential $V(\boldsymbol{\rho})$, which can be written as

$$\left(i\frac{\partial}{\partial\tau} - k_F^2\frac{1}{E_F}V(\boldsymbol{\rho}) + \nabla_\rho^2\right)\psi(\boldsymbol{\rho}, t) = 0\,. \qquad (3)$$

where $\tau = \frac{v_F}{2k_F}t = \frac{\hbar}{2m}t$ has also the dimension of a squared length. This equivalence has been noted in the seminal paper by DeRaedt et al. [22].

In quenched-disordered systems the disorder leads to a dependence of the quantities, which characterize the waves on the mode parameter E [39]. The mode statistics of these waves is described by the Bloch spectral density function [40] of the system, which is equal to the imaginary part $\mathsf{G}''(q, E)$ of the averaged one-particle Green's function, divided by π. In our case the relevant quantity, which becomes E dependent, is the deviation of the dielectric constant from its average value $\Delta(\boldsymbol{\rho})$. In an effective-medium description like the coherent-potential approximation (CPA) [7, 9, 39] this parameter is transformed as $\Delta(\boldsymbol{\rho}) \to \Sigma(s)$, where $\Sigma(s)$ is the so-called self energy and $s = E + i\eta\,;\, \eta \to +0$ is the complex modal parameter. In the low-disorder limit the CPA reduces to the self-consistent Born approximation [39], which can be derived independently within a nonlinear-σ-model description[12, 13, 36]. This description allows for going beyond the SCBA saddle point to calculate the unrenormalized wave diffusivity (see below).

Within the SCBA [12, 13, 36–38] the averaged single-particle Greens's function is given by

$$\mathsf{G}(q, s) \equiv \langle G(\mathbf{q}, s)\rangle = \frac{1}{-s - k_0^2\Sigma(s) + q^2} \qquad (4)$$

where $\mathbf{q}$ is the wavevector corresponding to the transverse spatial direction. As indicated above, the complex self-energy function $\Sigma(s) = \Sigma'(E) + i\Sigma''(E)$ encodes the influence of the disorder. In real (transverse) space we have

$$\begin{aligned}\mathsf{G}(\rho, s) \quad &= \quad -\tfrac{i}{4}H_0^{(1)}(k_\Sigma(s)\rho) \\ &\overset{\rho\to\infty}{\longrightarrow} \; -\tfrac{i}{4}\sqrt{\tfrac{2}{\pi k_\Sigma(s)\rho}}\,e^{ik_\Sigma(s)\rho}\end{aligned} \qquad (5)$$

where $H_0^{(1)}(x)$ is the Hankel function of first kind, and we have introduced the renormalized (complex) transverse wavenumber

$$k_\Sigma(s) = \sqrt{s + k_0^2\Sigma(s)} = k'_\Sigma(E) + ik''_\Sigma(E) \qquad (6)$$

Taking the modulus-square we extract the scattering mean-free path as

$$\frac{1}{\ell_0(E)} = 2k''_\Sigma(E) \approx \frac{k_0^2}{k'_\Sigma(E)}\Sigma''(E) \qquad (7)$$

where the approximation holds for $|\Sigma''| \ll |(E/k_0^2) + \Sigma'|$. The two-dimensional version of the SCBA equation for the self-energy has the form [12, 13, 36, 38]

$$\Sigma(s) = \frac{k_0^2}{2}\gamma\underbrace{\frac{4\pi}{q_c^2}\int_{|\mathbf{q}|<q_c}\frac{d^2\mathbf{q}}{(2\pi)^2}}_{\sum_{\mathbf{q}}^{q_c}}\mathsf{G}(q, s) \equiv \frac{k_0^2}{2}\gamma\mathsf{G}(s)\,, \qquad (8)$$

where $\gamma = \langle\Delta(\rho)^2\rangle$ is the disorder parameter. The cutoff parameter q_c is proportional to the inverse of the disorder correlation length (of the order of the diameter of the grains with different refraction indices). We have chosen the prefactor in such a way that $\sum_{\mathbf{q}}^{q_c} = 1$.

One can calculate from the SCBA an unrenormalized diffusivity, which describes the initial diffusive motion of the ray in the $x-y$ direction before the localization effects become dominant. This diffusivity is obtained by the Gaussian fluctuations of the self-energy field beyond the SCBA saddle point [12, 36, 41, 42]:

$$D_0(E) = -\epsilon\frac{1}{C(0, E)}\frac{\partial^2}{\partial q_x}C(\mathbf{q}, E)\bigg|_{\mathbf{q}=0)} \qquad (9)$$

with

$$C(\mathbf{q}, E) = \varphi(\mathbf{q}, E) - \frac{2}{\gamma} \qquad (10)$$

and

$$\varphi(\mathbf{q}, E) = k_0^4\sum_{\mathbf{k}}^{q_c}\mathsf{G}(\mathbf{k}+\tfrac{1}{2}\mathbf{q}, s_+)\mathsf{G}(\mathbf{k}-\tfrac{1}{2}\mathbf{q}, s_-) \qquad (11)$$

where $s_\pm = E \pm \epsilon$.

As shown in the supplemental material, using explicitly the SCBA equation, the unrenormalized diffusivity can be represented as

$$D_0(E) = \frac{\pi k_\Sigma^2}{q_c^2 k_0^2\Sigma''(E)\mathsf{G}''(E)} = k'_\Sigma(E)\ell_{\rm tr}(E) \qquad (12)$$

where we have introduced a *transport mean-free path* as

$$\frac{1}{\ell_{\rm tr}(E)} = \frac{\mathsf{G}''(E)q_c^2}{\pi}\frac{1}{\ell_0(E)} = \frac{\mathsf{G}''(E)q_c^2}{\pi}\frac{k_0^2}{k'_\Sigma(E)}\Sigma''(E) \qquad (13)$$

It should be remarked at this point that the "diffusivity" is dimensionless, because the "time" $\tau = z/2k_0$ has the dimension of a squared length.

We now use the self-consistent localization theory [9, 27–29] for calculating the *renormalized* diffusivity, which includes the localization effects.

We are interested in the averaged wave intensity, which includes the interference effects. This means that we cannot use the modulus-square of the averaged Green's function. Our object of interest is therefore the intensity Green's function $g(\mathbf{r}, \mathbf{r}', \tau, \tau') = \langle|G(\mathbf{r}, \mathbf{r}', \tau - \tau')|^2\rangle$ In Fourier space one has to use convolutions of the two Green's functions, which is conveniently formulated in terms of difference and center-of-mass coordinates and one obtains (with $\mathbf{q}$ and ω referring to the center-of-mass coordinates and E to $\tau-\tau'$):

$$\begin{aligned} g(q,\omega) &= \frac{1}{(2\pi)^3}\int dE \int d^2\mathbf{k}\Big\langle G(\mathbf{k}+\tfrac{1}{2}\mathbf{q}, s_+ + \tfrac{1}{2}\omega)G(\mathbf{k}-\tfrac{1}{2}\mathbf{q}, s_- - \tfrac{1}{2}\omega)\Big\rangle \\ &= \frac{1}{2\pi}\int dE\; g(E,q,\omega) \end{aligned} \tag{14}$$

with $s_\pm = E \pm i\epsilon$. The modal intensity Green's function $g(E,q,\omega)$ obeys a diffusion equation with a frequency-dependent modal "diffusivity" $D(\omega,E)$:

$$\Big(i\omega + q^2 D(\omega,E)\Big) g(E,q,\omega) = 1 \tag{15}$$

The Fourier transform of $D(\omega,E)$ is related to the *mean-square displacement* of the intensity

$$D(\tau,E) = \frac{d^2}{d\tau^2}R^2(\tau,E) \quad \Leftrightarrow \quad R^2(\omega,E) = \frac{D(\omega,E)}{(i\omega)^2} \tag{16}$$

In the renormalized localization theory of Vollhardt and Wölfle [27] $D(\omega)$ obeys the self-consistent relation

$$\frac{D_0(E)}{D(\omega,E)} = 1 + \frac{2}{\pi}\int_0^{1/\ell_{\rm tr}(E)} qdq \frac{1}{i\omega + q^2 D(\omega,E)}\,, \tag{17}$$

Eq. (17) can be put into the form

$$\frac{D(\omega,E)}{D_0(E)} = 1 - \frac{2}{\pi}\frac{1}{D_0(E)}\int_0^{1/\ell_{\rm tr}(E)} qdq \frac{1}{\frac{i\omega}{D(\omega,E)} + q^2} \tag{18}$$

In two dimensions one has always localization, which means that $D(\omega,E) = i\omega\xi^2(E)$, and the real part of $D(\omega,E)$ vanishes for small ω.

Taking the *real part* of both sides of Eq. (18) leads to an implicit equation for the localization length

$$1 = \frac{2}{\pi}\frac{1}{D_0(E)}\int_0^{1/\ell_{\rm tr}(E)} qdq \frac{1}{\frac{1}{\xi^2(E)} + q^2}\,, \tag{19}$$

which can be solved as

$$\begin{aligned} \xi(E) &= \ell_{\rm tr}(E)\big[e^{\pi D_0(E)} - 1\big]^{1/2} &\text{(20a)}\\ &\approx \ell_{\rm tr}(E) e^{\frac{\pi}{2}D_0(E)} \\ &= \ell_{\rm tr}(E) e^{\frac{\pi}{2}k'_\Sigma(E)\ell_{\rm tr}(E)} &\text{(20b)} \end{aligned}$$

This formula (with the unrenormalized wavenumber k_F) is also obtained from the scaling theory of localization [5, 6] and the so-called potential-well analogy [43–45], which is a simplified version of the theory of Vollhardt and Wölfle [27].

We are now interested in the *complex* diffusivity as a function of frequency. Combining Eqs. (17) and (19) we get

$$\frac{1}{D(\omega,E)} = \frac{1}{D_0(E)} + \frac{1}{i\omega\xi^2(E)} \tag{21}$$

Using Eq. (16) we obtain for the mean-square displacement

$$R^2(\omega,E) = \xi^2(E)\left(\frac{1}{i\omega} - \frac{1}{i\omega + \tau_\xi^{-1}(E)}\right) \tag{22}$$

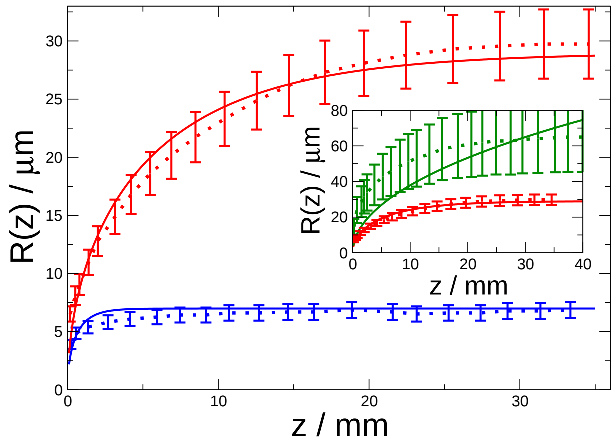

FIG. 1: Root-mean-square distance $R(z) = \sqrt{R^2(E,z)}$ with $E = 4/w_0^2 = 4/(3.3)^2 = 0.36$ as a function of the propagation distance $z = 2k_0\tau$ for $\lambda = 2\pi/k_0 = 633$ nm (top) and 405 nm (bottom) with index contrast $\Delta = \delta n/\langle n\rangle = 0.1/1.54 = 0.065$. The symbols are results of the simulation of Karbasi et al. [26], the lines are the results of the present theory.
Inset: Same as the main part with different index contrasts, namely $\Delta = \delta n/\langle n\rangle = 0.1/1.54 = 0.065$ (bottom) and $0.05/1.54 = 0.0325$ (top) for $\lambda = 633$ nm.

with the "localizion time"

$$\tau_\xi(E) = \frac{\xi^2(E)}{D_0(E)} \tag{23}$$

In the "time" domain we obtain

$$R^2(\tau,E) = \xi^2(E)\Big(1 - e^{-\tau/\tau_\xi(E)}\Big) \tag{24}$$

For $\tau \ll \tau_\xi(E)$ we have

$$R^2(\tau,E) = D_0(E)\tau \tag{25}$$

but near $\tau \approx \tau_\xi(E)$ the mean-square displacement levels off and becomes equal to $\xi^2(E)$.

For the diffusion propagator we obtain in a similar way

$$g(E,\mathbf{q},\tau) = \frac{1}{\xi^2(E)}\frac{1}{\frac{1}{\xi^2(E)} + q^2} + e^{-\tau/\tau_\xi(E)}e^{-D_0(E)q^2\tau} \tag{26}$$

For large "times" the diffusive contribution dies out (see also the discussion in the subsequent section) and we are left with the first localized contribution. In $x-y$ space this describes the typical fall-off of the intensity from its maximum at a particular site of a localized state, $\mathbf{r}_0$:

$$g(\boldsymbol{\rho}-\boldsymbol{\rho}_0, \tau=\infty) = \frac{1}{\xi^2(E)}\frac{1}{2\pi}\int_0^{1/\ell_{\rm tr}} qdq \frac{1}{\frac{1}{\xi^2(E)} + q^2} J_0(q|\boldsymbol{\rho}-\boldsymbol{\rho}_0|) \tag{27}$$

Taking Eq. (19) into account we get for the intensity at the center point $\mathbf{r}_0$

$$\begin{aligned} g(0,\tau=\infty) &= \frac{1}{4}\frac{D_0(E)}{\xi^2(E)} = \frac{1}{4\tau_\xi(E)} \\ &= \frac{q_c^2 k_0^2}{4\pi}\mathrm{G}''(E)\Sigma''(E)e^{-\pi D_0(E)} \end{aligned} \tag{28}$$

In summary, by means of Eqs. (7), (8), (12), (20b), (23), (24), (28) we are now able to calculate analytically the salient features of transverse localization in optical fibers as a function of the sample parameters. The most important quantity is the width of the beam $R(z)$ as a function of the propatation distance $z = 2k_0\tau$, which is determined by the unrenormalized diffusivity $D_0(E)$ and the localization length $\xi(E)$. The range of the possible values of the modal parameter $E = k_0^2\theta^2$ is determined by the numerical aperture $NA/\langle n\rangle\theta_{\max} = 2/k_0 w_0$, where w_0 is the width of the incipient Gaussian beam at the aperture. In the simulations of Karbasi et al. [24, 26] an initial beam with width parameter $w_0 = 3.3\mu^{-1}$ has been used. Therefore we took the value $E = 4/w_0^2 = 0.37\mu^{-2}$ for our calculations.

In Fig. 1 we compare the results for the channel width $R(\tau) = \sqrt{R^2(\tau)}$ of the simulations with the predictions of our theory for red ($\lambda = 0.633\mu$) and blue ($\lambda = 0.405\mu$) light. For the cutoff parameter we took $q_c = 6\mu^{-1}$, which corresponds to a correlation length of the order of 1 μ. The index contrast in the main body of the figure is the same as that in the simulation, namely $\delta n = 0.1$. In the inset the results of the simulation and theory for the index contrast $\delta n = 0.05$ are compared. We see that our theory describes the data in a satisfactory manner.

In conclusion we have combined an effective-medium theory for the mean-free path of light in a transversely disordered medium with the localization theory of Vollhardt and Wölfle [27]. With the help of this theory we have obtained analytical results for the channel width as a function of the propagation distance and the corresponding beam intensity. The results for the channel width agree well with those of the simulations based on the same paraxial wave equation. We believe that this analytical description will be very helpful for designing appropriate image transport devices for endoscopy.

The results reported here provide a deeper understanding of the transverse localization phenomena. We are now able to deal with transverse disorder in all systems which are well described in paraxial approximation, i.e. by a 2D plus time Schrödinger-like wave equation.

We think that the present analytic theory for transverse localization is therefore a milestone for the growing field of of disordered photonics, enabling the possibility to design a novel generation of disorder-based waveguides with potential applications in several fields ranging from endoscopy to mid-range optical communication.

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