Three-dimensional Anderson Localisation of Light in Materials with Fluctuating Electric and Magnetic Properties

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Abstract: Anderson localisation of electromagnetic waves, caused by disorder-induced arrest of wave diffusion, has been experimentally observed in systems with spatially fluctuating permeability, but only in lower dimensions, not in three dimensions. This paper introduces a novel theoretical approach to the Maxwell equations considering both electric and magnetic disorder. It demonstrates that when both the dielectric constant and magnetic permeability fluctuate in space, the spectral range for three-dimensional Anderson localisation significantly increases.

1. Introduction

Understanding the propagation and scattering of electromagnetic radiation in random media, especially visible light, is an issue, which is important in different parts of science [1–6]. A particularly interesting feature of waves in a disordered environment is the possibility of localisation, i.e. the absence of diffusion, demonstrated first for electron wave functions by Anderson [7]. Anderson localisation occurs with all kinds of waves, including atomic-matter and gravitational waves [6, 8–14].

Localisation of classical waves has first been discussed by John et al. [15, 16] for acoustical and later for electromagnetic waves (light) [17, 18]. The successful observation of weak localisation of light (the back-scattering cone) [19] created an impact for looking for strong Anderson localisation of light [4, 6, 20–23]. It was realised [24, 25] that the chances for the observation of this phenomenon are much higher in dimensionally reduced systems. This has been successfully demonstrated in paraxial structures with transverse (2-dimensional) disorder [26, 27] and two-dimensional photonic crystals [28]. In 3-dimensional media with a spatially fluctuating permittivity, however, until now, Anderson localisation has not been found [21, 21, 29–33]. Indeed, 3D localisation effects are often obscured by absorption or fluorescence processes, making its experimental demonstration extremely elusive [33].

It has even been demonstrated [34–36] that Anderson localisation of light in a three-dimensional system, containing a dense collection of resonant scatterers, does not occur due to a non-radiative (evanescent) coupling between the scatterers. The additional transport channel is associated with disorder-induced longitudinal fields [36, 37].

Recently, the possibility of obtaining Anderson localisation in 3D systems has been made plausible in numerical simulations of hyperuniform amorphous photonic materials [38, 39] and systems with overlapping spherical, perfectly conducting obstacles [40, 41].

On the other hand, the theoretical description of Anderson localisation of light is, until now, built on the ground of a mathematically questionable mapping of Maxwell's equations to

Anderson's Schrödinger equation of an electron in a random potential [17, 18]. This Schrödinger-equation analogy (called "potential-type approach" in Ref. [42]) which was taken over by the subsequent literature [4, 6, 25, 43–45], was shown [42]) to produce results, which were at variance with experiment: A wavelength dependence of the localisation length, predicted on base of this equation [25, 46], was not observed experimentally.

In the following treatment we start with formulating consistent eigenvalue equations for Maxwell's equations in the presence of a spatially varying electric permittivity $\epsilon(\mathbf{r})$ magnetic permeability $\mu(\mathbf{r})$. We then generalise the coherent-potential approximation (CPA) for this setting. In the fourth section we use the CPA results for the scattering mean-free path and the density of states for estimating the possibility of Anderson localisation using standard localisation theory. We find that by combining electric and magnetic disorder the chances for observing localisation of light in three dimensions are greatly enhanced with respect to the case where only one quantity ($\epsilon(\mathbf{r})$ *or* $\mu(\mathbf{r})$) is left to vary.

2. Eigenvalue equation for Maxwell's equations

It is a particular challenge to establish an adequate eigenvalue problem in the presence of disorder. As mentioned in the introduction, in most of the literature [4, 6, 18, 45, 47, 48], wave propagation in the presence of a spatially fluctuating permittivity $\epsilon(\mathbf{r}) = \langle\epsilon\rangle + \Delta\epsilon(\mathbf{r})$ is treated by establishing an analogy to the electronic Anderson problem. This is done in the following way: One first writes down the Helmholtz equation for the frequency-dependent electric field $\mathsf{E}(\mathbf{r}, \omega)$, as obtained from Maxwell's equations [49]:

$$\begin{aligned} \frac{\omega^2}{c_o^2}\frac{\epsilon(\mathbf{r})}{\epsilon_0}\mathsf{E}(\mathbf{r},\omega) &= \nabla\times[\nabla\times\mathsf{E}(\mathbf{r},\omega)] \\ &= -\nabla^2\mathsf{E}(\mathbf{r},\omega) + \nabla[\nabla\cdot\mathsf{E}(\mathbf{r},\omega)] \\ &\doteq \frac{\epsilon(\mathbf{r})}{\epsilon_0}\mathcal{L}_\mathsf{E}(\mathbf{r})\mathsf{E}(\mathbf{r},\omega) \end{aligned} \tag{1}$$

Here we introduced the linear operator $\mathcal{L}_\mathsf{E}(\mathbf{r}) = \frac{\epsilon_0}{\epsilon(\mathbf{r})}\nabla\times\nabla\times$, which we call Helmholtz operator for the electrical field. In terms of this operator Eq. (1) takes the form

$$\frac{\omega^2}{c_o^2}\mathsf{E}(\mathbf{r},\omega) = \mathcal{L}_\mathsf{E}(\mathbf{r})\mathsf{E}(\mathbf{r},\omega) \tag{2}$$

In the literature this equation is transformed as follows: The double curl is replaced by $-\nabla^2$. The second term $\nabla[\nabla\cdot\mathsf{E}(\mathbf{r},\omega)]$ is ignored [6, 18, 45, 47, 48], or projected out [4]. Then the coefficient of E on the LHS, which features the spectral parameter ω^2 of the eigenvalue equation, is rewritten as $\omega^2\epsilon_0 + \omega^2\Delta\epsilon(\mathbf{r})$. Finally, the second term is re-interpreted as an ω dependent potential $\mathcal{V}(\mathbf{r},\omega) = -\omega^2\Delta\epsilon(\mathbf{r})/\langle\epsilon\rangle$ [4, 6, 18, 45, 47, 48], i.e. the spectral parameter ω^2 enters into the "Hamiltonian" $\mathbf{H}(\mathbf{r},\omega) = -\nabla^2/\mu_0\langle\epsilon\rangle + \mathcal{V}(\mathbf{r},\omega)$ leading to the "Schrödinger equation"

$$\omega^2\mathsf{E}(\mathbf{r},\omega) = \mathbf{H}(\mathbf{r},\omega)\mathsf{E}(\mathbf{r},\omega) \tag{3}$$

This transformation of the electromagnetic Helmholtz equation (1) to a Schrödinger-like equation (3) was performed in order to be able to use the established theories for the Anderson localisation of non-interacting electrons [50–52] for Anderson localisation of light [6, 18, 47].

In two-dimensional systems this approach led to results, which were at variance with experiment: A wavelength dependence of the localisation length, predicted on base of Eq. (3) [25, 46], was not observed experimentally and is not predicted by a treatment, in which it is avoided that the spectral parameter ω^2 appears on the right-hand side of the Eigenvalue equation [42].

In the following we use versions of the electromagnetic Helmholtz eigenvalue problem, formulated without an ω dependent potential, in order to describe the influence of a spatially fluctuating electic permittivity $\epsilon(\mathbf{r})$ and magnetic permeability $\mu(\mathbf{r})$ on the electromagnetic spectrum. Such versions of the eigenvalue problem have been formulated in the literature for the electric [53], the magnetic [54] field, and the vector potential [55, 56]. As in these treatments.

We start by defining dimensionless electric and magnetic moduli $M_\epsilon(\mathbf{r}) := \epsilon_0/\epsilon(\mathbf{r})$ and $M_\mu(\mathbf{r}) := \mu_0/\mu(\mathbf{r})$. The generalisation of (1) for including magnetic disorder takes the form

$$\begin{aligned} \frac{\omega^2}{c_0^2}\mathsf{E}(\mathbf{r},\omega) &= M_\epsilon(\mathbf{r})\nabla\times[M_\mu(\mathbf{r})\nabla\times\mathsf{E}(\mathbf{r},\omega)] \\ &=: \mathcal{L}_\mathsf{E}\mathsf{E}(\mathbf{r},\omega)\,. \end{aligned} \tag{4}$$

The operator $\mathcal{L}_\mathsf{E}$ on the RHS of this equation is not Hermitian, if the ("naive") definition of the scalar product $<\mathsf{E}_1|\mathsf{E}_2>=\int d^3\mathbf{r}\mathsf{E}_1^*(\mathbf{r})\cdot\mathsf{E}_2(\mathbf{r})$ is used. Only if we define [57]

$$<\mathsf{E}_1|\mathsf{E}_2>:=\int d^3\mathbf{r}M_\epsilon^{-1}(\mathbf{r})\mathsf{E}_1^*(\mathbf{r})\cdot\mathsf{E}_2(\mathbf{r})\,, \tag{5}$$

the operator $\mathcal{L}_\mathsf{E}$ has the Hermitian property:

$$\begin{aligned} <\mathsf{E}_1|\mathcal{L}_\mathsf{E}\mathsf{E}_2> &= \int d^3\mathbf{r}\,\mathsf{E}_1^*(\mathbf{r})\cdot\left[\nabla\times M_\mu(\mathbf{r})[\nabla\times\mathsf{E}_2(\mathbf{r})]\right] \\ &= \int d^3\mathbf{r}\,M_\mu(\mathbf{r})\left[\nabla\times\mathsf{E}_1^*(\mathbf{r})\right]\cdot\left[\nabla\times\mathsf{E}_2(\mathbf{r})\right] \\ &= \int d^3\mathbf{r}\,\mathsf{E}_2(\mathbf{r})\cdot\left[\nabla\times M_\mu(\mathbf{r})[\nabla\times\mathsf{E}_1^*(\mathbf{r})]\right] \\ &\stackrel{!}{=} <\mathsf{E}_2|\mathcal{L}_\mathsf{E}\mathsf{E}_1>^*\ . \end{aligned} \tag{6}$$

The second line guarantees the positiveness of the spectrum. It is easily verified that for the scalar product without the fluctuating permittivity included, $\mathcal{L}_\mathsf{E}$ is not Hermitian, because extra terms involving ∇M_ϵ are obtained.

Similarly an equation for the magnetic field can be derived from Maxwell's equations

$$\begin{aligned} \frac{\omega^2}{c_0^2}\mathcal{H}(\mathbf{r},\omega) &= M_\mu(\mathbf{r})\nabla\times M_\epsilon(\mathbf{r})[\nabla\times\mathcal{H}(\mathbf{r},\omega)] \\ &=: \mathcal{L}_\mathcal{H}\mathcal{H}(\mathbf{r},\omega)\,. \end{aligned} \tag{7}$$

Here, the operator $\mathcal{L}_\mathcal{H}$ is Hermitian, if the scalar product includes a factor $M_\mu^{-1}(\mathbf{r})$. In the case of pure electric disorder ($M_\mu = const$) no special definition of the scalar product is needed. This (properly defined) eigenvalue equation for electric disorder was used recently for treating transverse two-dimensional Anderson localisation [42].

It is remarkable that for $\omega \neq 0$ Eqs. (4) and (7) automatically guarantee the transversality conditions

$$\nabla\cdot\left[\mathsf{E}(\mathbf{r},\omega)/M_\epsilon(\mathbf{r})\right]=0;\quad \nabla\cdot\left[\mathcal{H}(\mathbf{r},\omega)/M_\mu(\mathbf{r})\right]=0\,. \tag{8}$$

An equation, which is mathematically equivalent to Eq. (4) is obtained for the vector potential $\mathcal{A}(\mathbf{r},\omega)$, defined as $\nabla\times\mathcal{A}(\mathbf{r},\omega)=\mu(\mathbf{r})\mathcal{H}(\mathbf{r},\omega)$, if the Coulomb gauge $\nabla\cdot\mathcal{A}=0$ is applied, i.e. the scalar potential $\phi(\mathbf{r})$ is set equal to 0 [55, 58]. This equation then guarantees the transversality condition

$$\nabla\cdot\epsilon(\mathbf{r})\mathcal{A}(\mathbf{r})=0. \tag{9}$$

In order to formulate an analytic theory for the disorder-averaged physical quantities in a system described by (4) and (7) it is rather disadvantageous to work with the disorder dependent

scalar product. This can be avoided using symmetrised fields [53, 55, 59] $\widetilde{\mathbf{E}} := \mathsf{E}/\sqrt{M_\epsilon(\mathbf{r})}$ and $\widetilde{\mathbf{H}} := \mathcal{H}/\sqrt{M_\mu(\mathbf{r})}$ which obey the symmetrised Helmholtz equations

$$\begin{aligned}\frac{\omega^2}{c_0^2}\widetilde{\mathbf{E}}(\mathbf{r},\omega) &= M_\epsilon^{1/2}(\mathbf{r})\nabla\times M_\mu(\mathbf{r})[\nabla\times M_\epsilon^{1/2}(\mathbf{r})\widetilde{\mathbf{E}}(\mathbf{r},\omega)] \\ &=: \mathcal{L}_{\widetilde{\mathbf{E}}}\widetilde{\mathbf{E}}(\mathbf{r},\omega)\,, \end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\omega^2}{c_0^2}\widetilde{\mathbf{H}}(\mathbf{r},\omega) &= M_\mu^{1/2}(\mathbf{r})\nabla\times M_\epsilon(\mathbf{r})[\nabla\times M_\mu^{1/2}(\mathbf{r})\widetilde{\mathbf{H}}(\mathbf{r},\omega)] \\ &=: \mathcal{L}_{\widetilde{\mathbf{H}}}\widetilde{\mathbf{H}}(\mathbf{r},\omega)\,. \end{aligned} \tag{11}$$

Eqs. (10) and (11) now constitute conventional eigenvalue equations with operators $\mathcal{L}_{\widetilde{\mathbf{E}}}, \mathcal{L}_{\widetilde{\mathbf{H}}}$ that are Hermitian with respect to the scalar products $<\widetilde{\mathbf{E}}_1|\widetilde{\mathbf{E}}_2> = \int d^3\mathbf{r}\widetilde{\mathbf{E}}_1^*(\mathbf{r})\cdot\widetilde{\mathbf{E}}_2(\mathbf{r})$ and $<\widetilde{\mathbf{H}}_1|\widetilde{\mathbf{H}}_2> = \int d^3\mathbf{r}\widetilde{\mathbf{H}}_1^*(\mathbf{r})\cdot\widetilde{\mathbf{H}}_2(\mathbf{r})$.

In this transformed way the differential operators are manifestly Hermitian with respect to the conventional definition of the scalar product. In this form the eigenvalue problem can be dealt with in the usual way, using functional integrals and replica theory [51, 60].

3. Coherent-Potential approximation (CPA)

Generalising the derivation of Köhler *et al.* [60] we establish a coherent-potential approximation (CPA), based on Eqs. (10), (11), along the lines of our pevious work on elasticity [61].

The CPA arises as a saddle-point equation of an effective field theory, constructed by field-theoretic methods, see Ref. [60] and the Appendix. This variational derivation is equivalent to the traditional method [62] requiring that the scattering T matrix of the "perturbation" $M_{\alpha,i} - M_\alpha(z)$, $(\alpha = \epsilon, \mu)$ be zero on the average. In the CPA the disordered system is replaced by an effective medium, in which the fluctuating quantities (in our case $M_\epsilon(\mathbf{r})$ and $M_\mu(\mathbf{r})$) are replaced by uniform, but frequency-dependent, complex quantities $M_\epsilon(z)$ and $M_\mu(z)$, where $z = \frac{1}{c_0}\omega + i\eta$, ($\eta$ is an infinitesimal positive real number), except inside a cavity around the midpoint $\mathbf{r}_i$. The volume of the cavity is V_c, and in this region $M_{\epsilon,\mu}$ take their fluctuating values evaluated at $\mathbf{r}_i$ $M_{\epsilon,i} \doteq M_\epsilon(\mathbf{r}_i)$ and $M_{\mu,i} \doteq M_\mu(\mathbf{r}_i)$. Within CPA these quantities are assumed to be uncorrelated, which means that V_c must be larger than the correlation volume ξ^3, where ξ is the correlation length. This. naturally introduces an ultraviolet wavenumber cutoff $k_\xi \propto \xi^{-1}$ into the effective medium. In our treatment, this cutoff replaces the radius of the first Brillouin zone (in crystals) and the Debye cutoff (in glasses) for the definition of the scattering density of states $g(\omega)$ which samples the states relevant for the disorder scattering:

$$g(\omega) = 2\omega\rho(\lambda) = 2\omega\frac{1}{\pi}\mathrm{Im}\{G(z)\}\,, \tag{12}$$

where $\rho(\lambda)$ is the density of levels (eigenvalues), $G(z)$ is the local Green's function

$$G(z) = \frac{3}{k_\xi^3}\int_0^{k_\xi} dk k^2 G(k,z)\,, \tag{13}$$

and $G(k,z)$ is the wavenumber dependent Green's function of the effective medium

$$G(k,z) = \frac{1}{-z^2 + k^2 M_\varepsilon(z) M_\mu(z)}\,. \tag{14}$$

We emphasise that – in contrast to the treatment using the nonlinear-sigma-model theory [17, 18] - in CPA the small parameter for justifying the saddle-point approximation is not the relative

variance of the fluctuating quantities [51], but the ratio V_c/V between the cavity volume and the volume V of the sample [60]. This enables to treat the case of strong disorder, where the relative variance may take any value.

The CPA equations read [60]

$$0 = \left\langle \frac{M_{\epsilon,i} - M_\epsilon(z)}{1 + q\big(M_{\epsilon,i} - M_\epsilon(z)\big)\Lambda_\epsilon(z)} \right\rangle_\epsilon \tag{15}$$

and

$$0 = \left\langle \frac{M_{\mu,i} - M_\mu(z)}{1 + q\big(M_{\mu,i} - M_\mu(z)\big)\Lambda_\mu(z)} \right\rangle_\mu \tag{16}$$

with $q = V_c k_\xi^3/3\pi^2$. The parameter q must be smaller than 1 and can be interpreted as a mean-field critical percolation threshold [60]. Because the critical percolation threshold for 3-dimensional continuum percolation is around 0.3, we take $q = 0.3$ in the numerical calculations that we performed to show graphically the effect of the disorder.

The quantities $\Lambda_{\epsilon,\mu}(z)$ are defined by

$$\Lambda_{\epsilon,\mu}(z) = \frac{1}{M_{\epsilon,\mu}(z)}\big[1 + z^2 G(z)\big] \tag{17}$$

We note that the CPA equations (15) and (16) are completely symmetric with respect to ϵ and μ, i.e. they hold for both, Eqs (10) and (11). We further note that if the distributions of the two spatially fluctuating quantities are the same, $\mathbf{P}(M_{\epsilon,i}) = \mathbf{P}(M_{\mu,i})$, it results $M_\epsilon(z) = M_\mu(z)$. Therefore the CPA equations reduce to the ones one would obtain if one would take $M_\epsilon(\mathbf{r}) = M_\mu(\mathbf{r})$ from the outset.

The averages $\langle \ldots \rangle_{\epsilon,\mu}$ are to be performed with distribution densities $\mathbf{P}_\epsilon(M_{\epsilon,i})$ and $\mathbf{P}_\mu(M_{\mu,i})$. For our calculations, in order to be able to treat the case of strong disorder, we take log-normal distributions [60]

$$\mathbf{P}_\epsilon(M_{\epsilon,i}) = \mathbf{P}_{\ln}\left(\frac{M_{\epsilon,i}}{M_\epsilon^{(0)}}, \sigma_\epsilon\right) \qquad \mathbf{P}_\mu(M_{\mu,i}) = \mathbf{P}_{\ln}\left(\frac{M_{\mu,i}}{M_\mu^{(0)}}, \sigma_\mu\right) \tag{18}$$

with

$$\mathbf{P}_{\ln}(x,\sigma) = [\sqrt{2\pi}\sigma x]^{-1} e^{-\ln^2(x)/2\sigma^2}\,. \tag{19}$$

Here $M_\epsilon^{(0)}$ and $M_\mu^{(0)}$ are the medians of the moduli.

The relative variances of the two distributions $\gamma_\epsilon = \langle (M_\epsilon - \langle M_\epsilon \rangle)^2 \rangle / \langle M_\epsilon \rangle^2 = e^{\sigma_\epsilon^2} - 1$ and $\gamma_\mu = \langle (M_\mu - \langle M_\mu \rangle)^2 \rangle / \langle M_\mu \rangle^2 = e^{\sigma_\mu^2} - 1$ are the control parameters of the theory.

From the Green's function (14) we can read off the formula for the (scattering) mean-free path

$$\frac{1}{\ell(\omega)} = \frac{2\omega}{c_0}\mathrm{Im}\left\{\frac{1}{\big[M_\epsilon(z)M_\mu(z)\big]^{1/2}}\right\} \tag{20}$$

and the speed of light inside the medium [63]:

$$v(\omega) = c_0\ \mathrm{Re}\big\{\big[M_\epsilon(z)M_\mu(z)\big]^{1/2}\big\} \tag{21}$$

In turn, from these quantities we can calculate the frequency-dependent (unrenormalised) diffusivity

$$D_0(\omega) = \frac{1}{3} v(r\omega)\ell(\omega)\,. \tag{22}$$

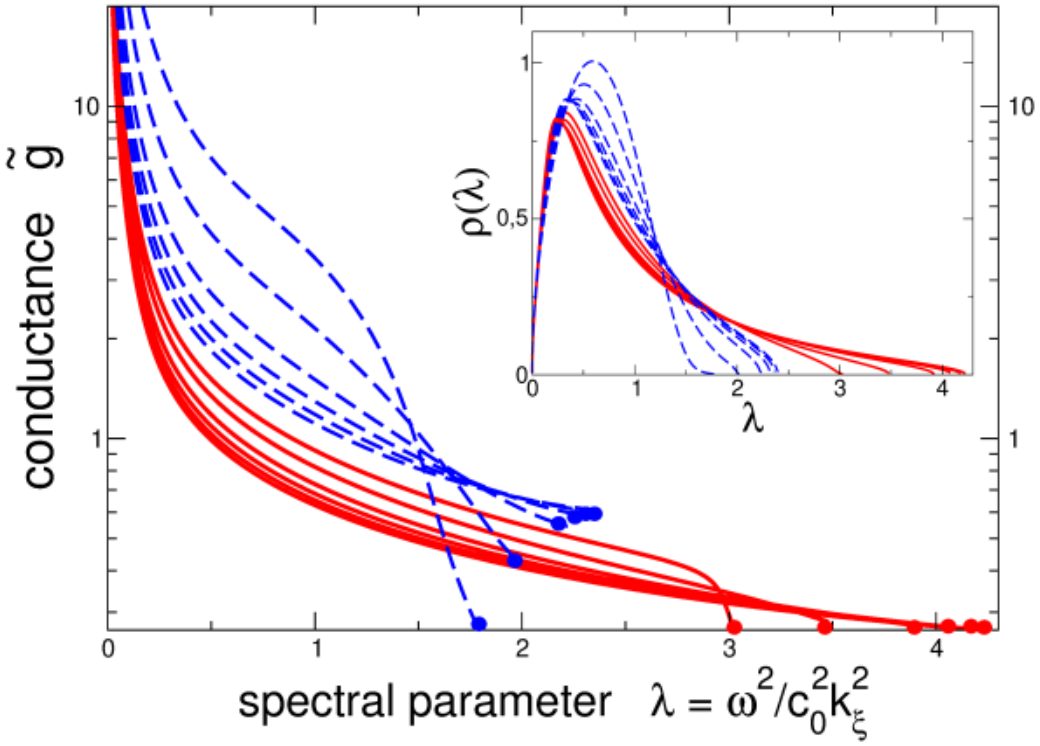

Fig. 1. "Conductance" $\tilde{g}(\omega) = g(\omega)D(\omega)$ against frequency, calculated in CPA for a log-normal distribution of M_ϵ and M_μ, Dashed blue lines: Only one quantity, say $M_\epsilon(\mathbf{r})$ is fluctuating, the relative variance $\gamma_\epsilon = e^{\sigma_\epsilon^2} - 1$ increases as $\gamma_\epsilon = 0.25$, 0.5, 1., 1.5, 2. 2.5.
Continuous red lines: Both quantities $M_\epsilon(\mathbf{r})$ and $M_\mu(\mathbf{r})$ are fluctuating, one of the variances, say, γ_ϵ is held fixed at 2.5, the other variance increases from $\gamma_\mu = 0.25$ to $\gamma_\mu = 2.5$. in steps as before.
Inset: density of eigenvalues $\rho(\lambda)$ for the same CPA calculations.
The full circles in the main panel mark the end of the spectrum, given by $\rho(\lambda)$ in the inset.

Before we use the CPA for estimating the localisation properties of disordered electromagnetic systems at finite frequency ω, we would like to comment on the limit $\omega \to 0$. As pointed out by Köhler *et al.* [60], in this limit the effective-medium expression of Bruggeman [64] for the permittivity of mixed dielectric materials is obtained. Contrary to this, the CPA applied to the potential-type treatment of Maxwell's equation [6], mentioned in the beginning, gives just the arithmetic average of the permittivity in the $\omega \to 0$ limit, because the non-trivial influence of the disorder in this approach is multiplied by ω^2 and just vanishes in the DC limit. This shows once more that a proper treatment of Maxwell's equations is necessary.

4. Anderson localisation

We now turn to the discussion of the impact of electrical and magnetic disorder on Anderson localisation of light. This phenomenon is known [8] to arise from interference of closed scattering paths. According to the self-consistent theory of Anderson localisation [11, 52, 65] in the version

used for classical waves [66–68] the renormalised diffusion coefficient, which includes the localisation phenomena, is given by [69]

$$D(\Omega,\omega) = D_0(\omega) - D(\Omega,\omega)P_0(\Omega,\omega) \tag{23}$$

Here Ω denotes the frequency corresponding to the diffusion dynamics of the radiation, and $P_0(\Omega,\omega)$ denotes the return probability

$$P_0(\Omega,\omega) = \frac{1}{\pi g(\omega)} \sum_{|\mathbf{q}|<q_0} \frac{1}{-i\Omega + q^2 D(\Omega,\omega)} \,. \tag{24}$$

The upper cutoff q_0 has been introduced, because the interference is only effective in the $\mathbf{q}$ region, where the diffusion approximation holds. In the original papers on electron localisation [11,52,65] the inverse mean-free path ℓ^{-1} has been taken for q_0, in the literature on phonon localisation [67,70] the Debye cutoff k_D, instead. Here we choose to take the correlation cutoff $q_0 = k_\xi$ as upper cutoff. The self-consistent Eq. (23) can now be written in the form

$$D(\Omega,\omega) = D_0(\omega) - \frac{3}{\pi k_\xi^3 g(\omega)} \int_0^{k_\xi} dq \frac{q^2}{q^2 - \frac{i\Omega}{D(\Omega,\omega)}} \tag{25}$$

Localisation or otherwise is now defined to occur if the quantity

$$\lim_{\Omega\to 0} D(\Omega,\omega) \tag{26}$$

vanishes or not.

We now assume that a frequency ω^* exists (mobility edge) which separates the extended states ($\omega < \omega^*$) from the localised ones ($\omega > \omega^*$). In the localised regime the quantity $-i\Omega/D(\Omega,\omega)$ becomes a real quantity, namely the square of the inverse localisation length. Right at the mobility edge $\omega = \omega^*$, this quantity becomes zero, and we have

$$D(\Omega,\omega) = \left[1 - \frac{3}{\pi k_\xi^2 g(\omega) D_0(\omega)}\right] \tag{27}$$

On the other hand, at the mobility edge, $D(\Omega,\omega) = 0$, so that the dimensionless quantity ("conductivity")

$$\tilde{g}(\omega) = k_\xi^2 g(\omega) D_0(\omega) \tag{28}$$

has to be equal to $3/\pi \approx 1$ at the mobility edge. Values of $\tilde{g}(\omega)$ larger than $\sim$1, therefore, lead to delocalisation, values smaller than $\sim$1 to localisation.

In Fig. 1 we have plotted this quantity, calculated in CPA against the dimensionless spectral parameter $\lambda = \omega^2/c_0^2 k_\xi^2$. We consider two scenarios:

(*i*) Only one of the moduli, say, $M_\epsilon(\mathbf{r})$ is considered to have spatial fluctuations with variance γ_ϵ increasing from 0.25 to 2.5: *electric (or magnetic) disorder only* (dashed blue lines).

(*ii*) Setting one of the variances, say, $\gamma_\mu = 2.5$ and increasing the other, γ_ϵ from 0.25 to 2.5: *Combined electric and magnetic disorder* (continuous red lines).

It is seen that in the case of the combined electric and magnetic disorder the values of $\widetilde{g}$ are (away from the vicinity of the band edge) much lower and, in particular, the *spectral range* for which $\widetilde{g}$ is smaller than $\sim$ 1 is much more extended. Fig. 1 comprises the central result of the present contribution.

In the case of one fluctuating quantity only (blue dashes) our results confirm the known fact that for classical waves localised states exist only within a narrow vicinity of the band edge [15, 18, 71]. This narrow range is – according to our estimate – largely extended in the combined case.

We therefore recommend for meeting the challenge of experimentally observing 3D Anderson localisation the consideration of disordered materials with both electric and magnetic disorder. Such materials could be e.g. polymeric materials with superparamagnetic inclusions [72].

Let us now discuss the recent numerical results of Yamilov *et al.* [40, 41] in the light of our findings. The authors considered two cases of systems with the disorder induced by overlapping spherical obstacles. These spheres were designed to have in the first case a high electric permittivity, in the second case perfect electric conduction inside the spheres. In their first system with high dielectric permittivity of the spheres they consider the case of electric disorder only. In agreement with our results they find no localisation. On the other hand, by using perfectly conducting obstacles they completely expel the time-varying electric and magnetic fields from the obstacles, just effectively introducing a combination of electric and magnetic disorder. Thus their numerical observation of Anderson localisation for the perfectly conducting obstacles corresponds to our prediction of localisation for the case of combined electric and magnetic disorder.

Summarising, we have presented a mean-field theory for combined electric and magnetic disorder based on eigenvalue equations derived from Maxwell's equations, which involve manifestly Hermitian operators. The results for the dimensionless conductance suggest systems with combined electric and magnetic disorder as candidates for 3D Anderson localisation.

Appendix: Derivation of the CPA

The 3×3 Green's function matrix $G_{\alpha\beta}(\mathbf{r}, \mathbf{r}', z)$, corresponding to (10) of the main text, obeys the matrix equation [73]

$$\sum_{\gamma} A_{\alpha\gamma}[M_\epsilon(\mathbf{r}), M_\mu(\mathbf{r})] G_{\gamma\beta}(\mathbf{r}, \mathbf{r}', z) = \delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r}') \tag{29}$$

with

$$A_{\alpha\gamma}[M_1, M_2] = \Big[- z^2 \delta_{\alpha\gamma} - c^2 M_\epsilon^{1/2}(\mathbf{r}) \nabla \times M_\mu(\mathbf{r}) \nabla \times M_\epsilon^{1/2}(\mathbf{r}) \Big]_{\alpha\gamma}$$

The complex frequency parameter is $z = \omega + i\eta$. For Eq. (6) The indices ϵ and μ have to interchanged.

We now apply functional-integral theory and represent the Green's function as a functional derivative of the following generating functional [74]

$$Z[J] \quad = \quad \int \mathcal{D}[u^*(\mathbf{r}) u(\mathbf{r})] e^{-(u|A|u) + (u|J|u)} \propto e^{-\mathrm{Tr}\{\ln[A-J]\}}$$

The trace operation includes a Cartesian trace and a diagonal spatial integration. The matrix elements of the inverse-Green's operator are

$$\begin{aligned} (u|A|u) \quad &= \quad \sum_{\alpha\beta} \int \int d^3\mathbf{r} d^3\mathbf{r}' u_\alpha^*(\mathbf{r}') \Big(A_{\alpha\beta}(z, \mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \Big) u_\beta(\mathbf{r}) \\ &= \quad \sum_{\alpha\beta} \int d^3\mathbf{r} u_\alpha^*(\mathbf{r}) A_{\alpha\beta}(z, \mathbf{r}) u(\mathbf{r})_\beta \,, \end{aligned} \tag{30}$$

those of the source fields are

$$(u|J|u) = \sum_{\alpha\beta} \int \int d^3\mathbf{r} d^3\mathbf{r}' u_\alpha^*(\mathbf{r}') J_{\alpha\beta}(\mathbf{r}\mathbf{r}') u_\beta(\mathbf{r}) \,, \tag{31}$$

and of the Green's functions

$$(J|G|J) = \sum_{\alpha\beta} \int \int d^3\mathbf{r} d^3\mathbf{r}' J^*_\alpha(\mathbf{r}') G_{\alpha\beta}(\mathbf{r}', \mathbf{r}) J_\beta(\mathbf{r}) \tag{32}$$

Consequently the Green's functions are given by the functional derivative

$$\begin{aligned} G_{\alpha\beta}(\mathbf{r}, \mathbf{r}') &= \frac{1}{Z(J=0)} \frac{\delta}{\delta J_{\alpha\beta}(\mathbf{r}, \mathbf{r}')} Z[J] \Big|_{J=0} \\ &= \frac{1}{Z(J=0)} \int \mathcal{D}[u^*_\alpha(\mathbf{r}), u_\alpha(\mathbf{r})] u_\alpha(\mathbf{r}) u_\beta(\mathbf{r}') e^{-(u|A|u)+(u|J|u)} \end{aligned} \tag{33}$$

In Replica theory [75] one effectively just discards the prefactor $\frac{1}{Z(J=0)}$ and works with the first of the n replicas. The result is

$$\langle G_{\alpha\beta}(\mathbf{r}, \mathbf{r}') \rangle = \frac{\delta}{\delta J(\mathbf{r}', \mathbf{r})} \langle Z[J] \rangle \Big|_{J=0} \propto \left\langle \int \mathcal{D}[u^*_\alpha(\mathbf{r}), u_\alpha(\mathbf{r})] u_\alpha(\mathbf{r}) u_\beta(\mathbf{r}') e^{-(u|A|u)} \right\rangle \tag{34}$$

where the average has to be taken with respect to the fluctuating quantities $M_\epsilon(\mathbf{r}) := M_{\epsilon,\mathbf{r}}$ and $M_\mu(\mathbf{r}) := M_{\mu,\mathbf{r}}$. As in Ref. [60] we now use a method (originally introduced by Faddeev and Popov for fixing the gauge in Yang-Mills theories [76]) introducing functional δ functions in order to replace the fluctuating quantities $M_{\epsilon,\mathbf{r}}$ and $M_{\mu,\mathbf{r}}$ by auxiliary fields $Q_\epsilon(\mathbf{r}, z)$ and $Q_\mu(\mathbf{r}, z)$ and represent this δ function, in turn, by a further integral using a second set of auxiliary fields ("ghost fields") $\Lambda_\epsilon(\mathbf{r}, z)$ and $\Lambda_\mu(\mathbf{r}, z)$:

$$\begin{aligned} Z[J] &= \int \mathcal{D}[u^*_\alpha, u_\alpha] e^{-(u|A[M_{\epsilon,\mathbf{r}}, M_{\mu,\mathbf{r}})]|u)} e^{(u|J|u)} \\ &= \int \mathcal{D}[u^*_\alpha, u_\alpha] \int \mathcal{D}\big[Q_\epsilon, Q_\mu\big] e^{-(u|A[Q_\epsilon, Q_\mu]|u)} e^{(u|J|u)} \delta(M_{\epsilon,\mathbf{r}} - Q_\epsilon) \delta(M_{\mu,\mathbf{r}} - Q_\mu) \\ &= \int \mathcal{D}[u^*_\alpha, u_\alpha] \int \mathcal{D}[Q_\epsilon, Q_\mu] e^{-(u|A[Q_\epsilon, Q_\mu]|u)} e^{(u|J|u)} \int \mathcal{D}[\Lambda_\epsilon, \Lambda_\mu] \, e^{-(\Lambda_\epsilon | M_{\epsilon,\mathbf{r}} - Q_\epsilon)} \, e^{-(\Lambda_\mu | M_{\mu,\mathbf{r}} - Q_\mu)} \\ &= \int \mathcal{D}\big[Q_\epsilon, Q_\mu\big] \int \mathcal{D}\big[\Lambda_\epsilon, \Lambda_\mu\big] e^{-\operatorname{tr}\left\{ \ln\left[A[Q_\epsilon, Q_\mu - J \right] \right\}} \, e^{-(\Lambda_\epsilon | M_{\epsilon,\mathbf{r}} - Q_\epsilon)} \, e^{-(\Lambda_\mu | M_{\mu,\mathbf{r}} - Q_\mu)} \end{aligned} \tag{35}$$

where $A[Q_\epsilon, Q_\mu]$ is now the operator $A[M_{\epsilon,\mathbf{r}}, M_{\mu,\mathbf{r}}]$ with the M's replaced by the Q's. The last line results from integrating out the original fields $u^\alpha(\mathbf{r})$.

As in [60] we now devise the following coarse-graining procedure:

- We tile the total space into N_c cells of (approximate) volume $V_c = V/N_c$, where $V = L^3$ is the total volume.
- Within a cell with label i we replace the moduli by their average in each cell and assume that the quantities M_ϵ and M_μ do not fluctuate anymore inside the cell.
- The fluctuations of $M_{\epsilon,i}$ and $M_{\mu,i}$ and are assumed to be uncorrelated, i.e.

$$P(M_{\epsilon,1} \ldots M_{\epsilon,i}) = \prod_i p(M_{\epsilon,i}) \,, \tag{36}$$

$$P(M_{\mu,1} \ldots M_{\mu,i}) = \prod_i p(M_{\mu,i}) \,. \tag{37}$$

- The assumption of uncorrelated fluctuations implies that the volume of the cells must be larger or at least equal to the correlation length ξ of the fluctuations of $M_{\epsilon,i}$ and $M_{\mu,i}$, which is assumed to be the same. Consequently, the $\mathbf{k}$ summations in the subsequent analysis are confined to an ultraviolett cutoff $k_\xi = \nu/\xi$, where ν is of the order of unity.

The matrix elements in the exponents of the two last factors in the bottom line of Eq. (35) take the form

$$(\Lambda_\epsilon | M_{\epsilon,\mathbf{r}} - Q_\epsilon) \to \frac{V_c}{V} \sum_i \Lambda_{\epsilon,i} \Big(M_{\epsilon,i} - Q_{\epsilon,i} \Big), \tag{38}$$

and the analogous expression holds for $\Lambda_\mu | M_{\mu,\mathbf{r}} - Q_\mu)$. We now start to evaluate the configurational average. exponentials in (35) have to be averaged with the distribution densities $p(M_{\epsilon_i})$ and $p(M_{\mu_i})$, and they factorise obviously. For the first averaged exponential we may write

$$\begin{aligned} \left\langle e^{-(\Lambda_\epsilon | M_\epsilon - Q_\epsilon)} \right\rangle &= \prod_{i=1}^{N_c} \left\langle e^{-\frac{V_c}{V} \Lambda_{i,\epsilon} (M_{i,\epsilon} - Q_{i,\epsilon})} \right\rangle_i \\ &= e^{\frac{V}{V_c} \ln \left(\left\langle \exp \left[-\frac{V_c}{V} \Lambda_{i,\epsilon} (M_{i,\epsilon} - Q_{i,\epsilon}) \right] \right\rangle_\mu , \right)} \end{aligned} \tag{39}$$

where $\langle \dots \rangle_\epsilon$ denotes an average with $p(M_{\epsilon,i})$. The factor $V/V_c = N_c$ comes from N_c multiplications of the same quantity $e^{\ln(\dots)}$. The second averaged exponential has the same form with ϵ being replaced by μ. Note that the two occurring volume ratios do not cancel each other due to the average inside the logarithm. Using (39) the average of the generating functional (35) can now be written as

$$\langle Z[J] \rangle = \int \mathcal{D}[Q, \Lambda] \, e^{-S_{\text{eff}}[Q_\epsilon, Q_\mu, \Lambda_\epsilon, \Lambda_\mu, J]} \tag{40}$$

with the effective action

$$\begin{aligned} S_{\text{eff}}[Q_\epsilon, Q_\mu, \Lambda_\epsilon, \Lambda_\mu, J] \;=\; & \text{tr}\{ \ln \left(A[Q_\epsilon, Q_\mu] - J \right) \} \\ & - \frac{V}{V_c} \ln \left(\left\langle e^{-\frac{V_c}{V} \Lambda_i (M_{\epsilon,i} - Q_\epsilon)} \right\rangle_\epsilon \right) - \frac{V}{V_c} \ln \left(\left\langle e^{-\frac{V_c}{V} \Lambda_i (M_{\mu,i} - Q_\mu)} \right\rangle_\mu \right) \end{aligned} \tag{41}$$

We now use the prefactor $\frac{V}{V_c} \gg 1$ of the action as large parameter for the saddle-point approximation.

As we are looking for a homogeneous saddle point of the action we replace the auxiliary fields by space-independent (but frequency-dependent) ones $Q_\epsilon(z), Q_\mu(z), \Lambda_\epsilon(z), \Lambda_\mu(z)$, which define frequency-dependent moduli and susceptibilities in a homogeneous effective medium. In this medium the average Green's functions depend only on $\mathbf{r} - \mathbf{r}'$ and so do the A matrices and the source fields J. The fields in the effective medium are completely transverse. In $\mathbf{k}$ space A is diagonal and has two identical entries ($A_{zz} = 0$), given by

$$A_{xx}(k,z) = A_{yy}(k,z) = -z^2 + k^2 Q_\epsilon(z) Q_\mu(z) := A(k,z) \tag{42}$$

G is also diagonal with the two entries

$$G_{xx}(k,z) = G_{yy}(k,z) = -\frac{\delta}{\delta J(\mathbf{k})} S_{\text{eff}} \Big|_{J=0} = \frac{1}{-z^2 + k^2 Q_\epsilon(z) Q_\mu(z)} := G(k,z) \,. \tag{43}$$

with $J(k) = J_{xx}(k)$. We can define a mean-field or effective-medium action as

$$\widetilde{S}_{eff}[Q_\epsilon, Q_\mu, \Lambda_\epsilon, \Lambda_m u, J{=}0] = 2 \sum_{\mathbf{k}} \{ \ln \left(A(k,z) \right) \} \tag{44}$$

$$\begin{aligned} & - \frac{V}{V_c} \ln \left(\left\langle e^{-\frac{V_c}{V} \Lambda_\epsilon (M_{\epsilon,i} - Q_\epsilon)} \right\rangle_\epsilon \right) - \frac{V}{V_c} \ln \left(\left\langle e^{-\frac{V_c}{V} \Lambda_\mu (M_{\mu,i} - Q_\mu)} \right\rangle_\mu \right) \\ & := \widetilde{S}_{\text{med}}[Q_\epsilon, Q_\mu] + \widetilde{S}_\epsilon[Q_\epsilon, \Lambda_\epsilon] + \widetilde{S}_\mu[Q_\mu, \Lambda_\mu] \end{aligned} \tag{45}$$

Varying the action with respect to $\Lambda_\epsilon(z)$ we obtain

$$\frac{\partial \widetilde{S}_{\text{eff}}}{\partial \Lambda_\epsilon} = \frac{\partial \widetilde{S}_\epsilon}{\partial \Lambda_\epsilon} = 0 \quad = \quad \frac{\left\langle -\frac{V_c}{V}\Lambda_\epsilon(z)(M_{\epsilon,i} - Q_\epsilon(z)) e^{-\frac{V_c}{V}\Lambda_\epsilon(z)[M_{\epsilon,i}-Q_\epsilon(z)]} \right\rangle_\epsilon}{\left\langle e^{-\frac{V_c}{V}\Lambda_\epsilon(z)[M_{\epsilon,i}-Q_\epsilon(z)]} \right\rangle_\epsilon}$$

$$\Rightarrow \qquad 0 \quad = \quad \left\langle \frac{M_{\epsilon,i} - Q_\epsilon(z)}{\exp\left[\frac{V_c}{V}\Lambda_\epsilon(s)\big(M_{\epsilon,i} - Q_\epsilon(z)\big)\right]} \right\rangle_\epsilon \tag{46}$$

Since $\frac{V_c}{V} \ll 1$ the exponential in the denominator can be expanded to first order

$$0 = \left\langle \frac{M_{\epsilon,i} - Q_\epsilon(z)}{1 + \frac{V_c}{V}\big(M_{\epsilon,i} - Q_\epsilon(z)\big)\Lambda_\epsilon(z)} \right\rangle_\epsilon \tag{47}$$

which is the CPA equation for $M_\epsilon(z)$. Similarly we cet the equation for M_μ by varying the action with respect to Λ_μ

$$0 = \left\langle \frac{M_{\epsilon,i} - Q_\epsilon(z)}{1 + \frac{V_c}{V}\big(M_{\epsilon,i} - Q_\epsilon(z)\big)\Lambda_\epsilon(z)} \right\rangle_\epsilon \tag{48}$$

Varying the action with respect to M_ϵ we get

$$\Lambda_\epsilon(z) = 2Q_\mu(z) \sum_{\mathbf{k}} \frac{k^2}{-z^2 + k^2 Q_\epsilon(z) Q_\mu(z)} \tag{49}$$

and similarly

$$\Lambda_\mu(z) = 2Q_\epsilon(z) \sum_{\mathbf{k}} \frac{k^2}{z + k^2 Q_\epsilon(z) Q_\mu(z)} \tag{50}$$

We define normalised susceptibility functions

$$\widetilde{\Lambda}_\epsilon(z) = \frac{1}{q}\frac{V_c}{V}\Lambda_\epsilon(z) = \frac{3}{k_\xi^3} Q_\mu(z) \int_0^{k_\xi} \frac{k^4}{z + k^2 Q_\epsilon(z) Q_\mu(z)} = \frac{1}{Q_\epsilon(z)}\big[1 + z^2 G(z)\big] \tag{51}$$

$$\widetilde{\Lambda}_\mu(z) = \frac{1}{q}\frac{V_c}{V}\Lambda_\mu(z) = \frac{3}{k_\xi^3} Q_\epsilon(z) \int_0^{k_\xi} \frac{k^4}{z + k^2 Q_\epsilon(z) Q_\mu(z)} = \frac{1}{Q_\mu(z)}\big[1 + z^2 G(z)\big] \tag{52}$$

with $q = V_c k_\xi^3 / 3\pi^2$. q can be related to a critical percolation threshold and therefore must be smaller than unity [60]. In our calculations we take $q = 0.3$. $G(z)$ is the normalised local Green's function

$$G(z) = \frac{3}{k_\xi^3} \int_0^{k_\xi} \frac{k^2}{z + k^2 Q_\epsilon(z) Q_\mu(z)} \tag{53}$$

from which the density of states can be calculated as

$$g(\omega) = \frac{2\omega}{\pi} \text{Im}\big\{G(z)\big\} \tag{54}$$

We finally obtain the following CPA equations

$$0 = \left\langle \frac{M_{\epsilon,i} - Q_\epsilon(z)}{1 + q\big(M_{\epsilon,i} - Q_\epsilon(z)\big)\widetilde{\Lambda}_\epsilon(z)} \right\rangle_\epsilon \tag{55}$$

which is the CPA equation for $M_\epsilon(z)$. Similarly we get the equation for M_μ by varying the action with respect to Λ_μ

$$0 = \left\langle \frac{M_{\mu,i} - Q_\mu(z)}{1 + q\big(M_{\mu,i} - Q_\mu(z)\big)\widetilde{\Lambda}_\mu(z)} \right\rangle_\mu \tag{56}$$

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