

Chapter 5

Critical Phenomena and Scaling Hypothesis

5.1 Scaling relations

Summary of some critical exponents

(Quantity)		2D Ising	3D Ising	Ising	Mean-field	
					Tricritical point	Lifshitz point
(Specific heat)	α	0	0.1	0	1/2	0
(Order parameter)	β	1/8	0.33	1/2	1/4	1/2
(Susceptibility)	γ	7/4	1.24	1	1	1
(OP $\leftrightarrow$ field at T_c)	δ	15	4.8	3		
(Correlation length)	ν	1	0.63	1/2	1/2	1/4
(Correlations at T_c)	η	1/4	0.04	0	0	2
(Upper critical dimension)	d_c			4	3	8

Question: Are these exponents really independent?

Closer look: Some general relations seem to be fulfilled:

$$\begin{aligned}
 \text{e.g.: } & \beta + \gamma = \beta\delta \\
 & \alpha + 2\beta + \gamma = 2 \quad (\text{Rushbrooke law}) \\
 2 - \alpha = & \begin{cases} \nu d & : d < d_c \quad (\text{Josephson law}) \\ \nu d_c & : d \geq d_c \quad (\text{Mean-field case}) \end{cases} \\
 & \gamma = \nu (2 - \eta)
 \end{aligned}$$

$\leadsto$ Scaling relations

History: Main discoveries around 1963

- Rushbrooke: Thermodynamic stability $\Rightarrow$ inequality $\alpha + 2\beta + \gamma \geq 2$
- Essam, Fisher: Numerical results suggest $\alpha + 2\beta + \gamma = 2$
(for arbitrary models and dimensions)
- Widom: Scaling hypothesis

5.2 Widom's scaling hypothesis

Provides a framework that helps to classify and "derive" scaling relations.

First introduced in a heuristic manner. A more formal derivation will be provided in chapter 6.

Of great practical use

Scaling analyses are fast ("back of the envelope" calculations).

Can be applied to a variety of problems/systems

(e.g., finite size scaling, polymers, dynamical systems, ...)

Often give correct relations without lengthy calculations

Here: "Static scaling hypothesis" for Ising-type systems

(a) Scaling hypothesis for the order parameter (Widom, 1963)

Order parameter is a function of $t = (T - T_c)/T_c$ and $h = H/k_B T$

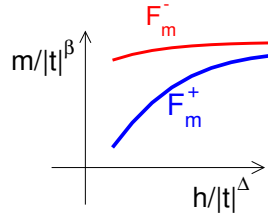
$$\text{We know: } m(t, h \rightarrow 0) = \begin{cases} 0 & : t > 0 \\ \pm A |t|^\beta & : t < 0 \end{cases} \text{ and } m(t = 0, h) = \pm B |h|^{1/\delta}$$

Question: Can one combine both formulae?

$$\text{Widom's Ansatz: } m(t, h) = \begin{cases} t^\beta F_m^+(h/t^\Delta) & : t > 0 \\ (-t)^\beta F_m^-(h/(-t)^\Delta) & : t < 0 \end{cases}$$

β, Δ : Universal exponents (Δ : "gap" exponent")

$F_m^\pm$: Scaling functions:



Data for different fields H should lie on the same curve
 $\leadsto$ "data collapse"

(NB: In order to implement this type of plot, one must either know or fit β , Δ , and T_c !)

To derive relations between exponents, consider limiting cases

- $h \rightarrow 0, t \neq 0$ ($|t|$ small)

$$m(t, h \rightarrow 0) = |t|^\beta F_m^\pm(h/|t|^\Delta) \stackrel{!}{=} \begin{cases} 0 & : t > 0 \\ \pm A |t|^\beta & : t < 0 \end{cases}$$

$$\Rightarrow F_m^+ = 0, \quad F_m^-(0) = \pm A \neq 0 \text{ (finite)}$$

$$\chi(h \rightarrow 0, t) = \frac{\partial m}{\partial h} \Big|_{h \rightarrow 0} = \frac{|t|^\beta}{|t|^\Delta} \frac{d}{dx} F_m^\pm(x) \Big|_{x \rightarrow 0} = |t|^{\beta-\Delta} F_m^{\pm'}(0) \stackrel{!}{\propto} |t|^{-\gamma}$$

$$\Rightarrow \gamma = \Delta - \beta, \text{ quad i.e., } \Delta = \beta + \gamma$$

- $t \rightarrow 0, h \neq 0$ ($|h|$ small)

$$m(t \rightarrow 0, h) = |t|^\beta F_m^\pm(h/|t|^\Delta) \stackrel{!}{\propto} |h|^{1/\delta} \text{ independent of } t!$$

$$\Rightarrow \text{Since } \Delta > 0: F_m^\pm(x) \sim |x|^\lambda \text{ for } x \rightarrow \infty$$

where λ has to be chosen such that $|t|^\beta |h/|t|^\Delta|^\lambda$ is independent of $|t| \Rightarrow \beta - \Delta\lambda = 0$, i.e., $\lambda = \beta/\Delta$

$$\begin{aligned} \text{Insert: } m(t \rightarrow 0, h) &\sim |h|^\lambda = |h|^{\beta/\Delta} \stackrel{!}{\propto} |h|^{1/\delta} \\ &\leadsto \beta/\Delta = 1/\delta \quad \Rightarrow \quad \Delta = \beta \delta \end{aligned}$$

$$\text{Together: } \boxed{\Delta = \gamma + \beta = \beta \delta}$$

Thus we have determined the gap exponent Δ and deduced a scaling relation!

Derivation of other scaling relations is more difficult, since the exponents α, ν, η do not appear in the critical behavior of the order parameter $\leadsto$ Must find other, similar scaling hypotheses!

(b) Scaling hypothesis for the free energy

Alternative to (a): Write singular part of the free energy density as

$$\boxed{f_s(t, h) = |t|^{2-\alpha} F_f(h/|t|^\Delta)}$$

$$\begin{aligned} \Rightarrow \text{Specific heat: } c &\sim \frac{\partial^2 f_s}{\partial t^2} \sim |t|^{-\alpha} F_f(0) + 0 \quad \checkmark \\ \text{Order parameter: } m &\sim \frac{\partial f_s}{\partial h} \sim |t|^{2-\alpha-\Delta} F'_f(h/|t|^\Delta) \stackrel{!}{\sim} |t|^\beta \\ &\Rightarrow 2 - \alpha - \Delta = \beta \\ \text{Susceptibility: } \chi &\sim \frac{\partial m}{\partial h} \sim |t|^{2-\alpha-2\Delta} F''_f(h/|t|^\Delta) \stackrel{!}{\sim} |t|^{-\gamma} \\ &\Rightarrow 2 - \alpha - 2\Delta = -\gamma \end{aligned}$$

Taking everything together, we obtain again $\Delta = \beta + \gamma$
and an additional scaling relation $\boxed{\alpha + 2\beta + \gamma = 2}$

NB: The relation $m \sim |t|^{2-\alpha-\Delta} F'_f(h/|t|^\Delta)$ also reproduces the scaling hypothesis (a) for the order parameter!

(c) Scaling hypothesis for the correlation function

In the same spirit, one also constructs a scaling form for the two-point correlation function $G(\vec{r}) = \langle m(\vec{r}_0) m(\vec{r}_0 + \vec{r}) \rangle - \langle m \rangle^2$.

$$\boxed{G(\vec{r}, t, h) = \frac{1}{r^{d-2+\eta}} F_G^\pm(r|t|^\nu, h/|t|^\Delta)}$$

This yields scaling relations for η and ν

$$\begin{aligned} \text{Example: } \chi|_{h=0} &\sim \int d^d r G(\vec{r}, t, h) \sim \int dr r^{1-\eta} F_G^\pm(r|t|^\nu, 0) \\ &\sim |t|^{-(2-\eta)\nu} \int dy y^{1-\eta} F_G^\pm(y) \stackrel{!}{\propto} |t|^{-\gamma} \\ \Rightarrow \quad &\boxed{\gamma = \nu(2 - \eta)} \end{aligned}$$

5.3 Scaling hypothesis and dimensional analysis

Question: How can we physically motivate the scaling hypothesis?

Rough answer (not quite correct, see Section 5.4):

At $T \rightarrow T_c$, the correlations length ξ diverges with $\xi \sim |t|^{-\nu}$. This divergence drives all other singularities: Close to T_c , the correlation length is the only relevant length scale.

$\leadsto$ "Dimensional analysis"

★ Dimensions:

- Singular part of the free energy density: $[f_s] = [\text{length}]^{-d} = [\xi]^{-d}$
- Order parameter density: $[m] = [\xi]^{d_m}$
(d_m is an independent exponent!)
- Conjugate field h : $[h] \cdot [m] = [f_s] \Rightarrow [h] = [\xi]^{-d-d_m}$

★ Scaling functions: Composed of "dimensionless" quantities!

- Free energy density: $f_s(t, h) \sim \xi^{-d} F_f(h \xi^{d+d_m})$
Specifically $h = 0$: $f_s \sim \xi^{-d} \stackrel{!}{\sim} |t|^{2-\alpha}$
With $\xi \sim |t|^{-\nu}$, one obtains the Josephson relation: $2 - \alpha = d\nu$
- Order parameter density: $m(t, h) \sim \xi^{d_m} F_m(h \xi^{d+d_m})$
Specifically $h = 0$: $m \sim \xi^{d_m} \sim |t|^\beta \Rightarrow d_m = \beta/\nu$
Generally: $m \stackrel{!}{\sim} |t|^\beta F_m(h/|t|^\Delta)$
 $\Rightarrow \Delta = \nu(d + d_m) = \nu d - \beta = 2 - \alpha - \beta$
Also, following 5.2 a): $\Delta = \gamma + \beta = \beta \delta \Rightarrow \alpha + 2\beta + \gamma = 2$
- Correlation functions in Fourier space:
 $G(\vec{k}) = \int d^d r G(\vec{r}) e^{i\vec{k} \cdot \vec{r}}$ with $G(\vec{r} - \vec{r}') = \langle m(\vec{r}) m(\vec{r}') \rangle - \langle m \rangle^2$
Dimension: $[G(\vec{r})] : [m]^2 = [\xi]^{2d_m} = [\xi]^{-2\beta/\nu}$
 $[G(\vec{k})] : [m]^2 [\text{length}]^d = [\xi]^{2d_m+d} = [\xi]^{d-2\beta/\nu}$
 $[k] : [\text{length}]^{-1} = [\xi]^{-1}$
 $\leadsto$ Scaling Ansatz: $G(\vec{k}) \sim \xi^{d-2\beta/\nu} F_G(k\xi)$
Specifically $\xi \xrightarrow{|t| \rightarrow 0} \infty$: Result should not depend on ξ !
 $\Rightarrow F_G(x) \stackrel{x \rightarrow \infty}{\sim} x^{2\beta/\nu-d} \Rightarrow G(\vec{k}) \stackrel{\xi \rightarrow \infty}{\sim} k^{2\beta/\nu-d} \stackrel{!}{\propto} k^{-2+\eta}$
 $\Rightarrow 2 - \eta = d - 2\beta/\nu = \frac{1}{\nu}(d\nu - 2\beta) = \frac{1}{\nu}(2 - \alpha - 2\beta) = \gamma/\nu$
 $\Rightarrow \boxed{\gamma = \nu(2 - \eta)}$

Conclusion: General recipe to construct scaling forms

- Find dimensions of the (dependent and independent) variables
- Construct scaling form from "dimensionless" quantities

In practice very successful. However, unfortunately, does not always work.

Problems that are already apparent at this point:

- Relations that contain the spatial dimension d (e.g., the Josephson relation) are obviously no longer valid at $d > d_c$ (in the mean-field regime). So what is going on here ?
- It is not clear a priori, why the dimension d_m should be independent of the other dimensions. (Not at all obvious, see next section!)

5.4 Influence of additional length scales

Preliminary consideration:

Rescale singular part of the Ginzburg-Landau free energy

$$\beta \mathcal{F}_{\text{sing}} = \int d^d r \left(\frac{1}{2} g (\nabla m)^2 + \frac{1}{2} b m^2 + \frac{1}{4} c m^4 \right) =: \int d^d r f_s$$

Rescale: $\Phi = \sqrt{g} m$ and set $r_0 = b/g \propto (T - T_c)$, $u_0 = c/g^2$

$$\Rightarrow \boxed{\beta \mathcal{F}_{\text{sing}} = \int d^d r \left(\frac{1}{2} (\nabla \Phi)^2 + \frac{1}{2} r_0 m^2 + \frac{1}{4} u_0 \Phi^4 \right)}$$

$\leadsto$ " Φ^4 theory": Typical starting point for field theories of Ising type phase transitions More generally – one of the favorite toy models in statistical field theory and quantum field theory

Dimensional analysis: $\beta \mathcal{F}_{\text{sing}}$ dimensionless

$$\begin{aligned} \rightarrow [f_s] &= [\text{length}]^{-d}, & [\Phi] &= [\text{length}]^{1-d/2} \\ [r_0] &= [\text{length}]^{-2}, & [u_0] &= [\text{length}]^{d-4} \end{aligned}$$

Remark: If the correlation length ξ is the only length that matters, then one would always have

$$\begin{aligned} r_0 \sim \xi^{-2} &\Rightarrow \xi \sim (T - T_c)^{-1/2} \Rightarrow \nu = 1/2 \\ f_s \sim \xi^{-d} &\Rightarrow (2 - \alpha) = d\nu \end{aligned}$$

However, in fact, neither $\nu = 1/2$ nor $(2 - \alpha) = d\nu$ are always valid, and both together are only valid at $d = d_c = 4$!

Question: Where does the argument go wrong?

Answer: The correlation length is not the only length scale that matters!

In addition, other length scales come in: Microscopic length scales and a mean-field length scale. Depending on the spatial dimension, one of the two modifies the critical exponents.

5.4.1 Microscopic length scales and anomalous dimensions

★ Correlation length

As shown before: If the correlation length is the only relevant length scales, one would have $\xi \sim 1/\sqrt{r_0} \sim (T - T_c)^{-1/2} \Rightarrow \nu = 1/2$

Now assume, a second length scale $l_0 \ll \xi$ exists

$$\rightarrow \text{New scaling Ansatz for } \xi: \xi \sim \frac{1}{\sqrt{r_0}} f_\xi(\underbrace{r_0 l_0^2}_{\text{dimensionless}})$$

Assume further power law behavior: $\lim_{x \rightarrow 0} f_\xi(x) \sim x^\theta$

$$\rightarrow \text{For } r_0 \rightarrow 0, \text{ one has: } \xi \sim r_0^{-1/2+\theta} \sim (T - T_c)^{-1/2+\theta} \Rightarrow$$

$$\boxed{\nu = 1/2 - \theta}$$

$\Rightarrow$ Introduction of an "anomalous dimension" θ solves our problem:
Exponent θ may differ from $1/2$.

Important: In order to introduce θ , we needed to assume the existence of a microscopic length l_0 . This is an unusual thought in the theory of critical phenomena. However, the exponent θ turns out to be independent of the actual value of l_0 !

★ Correlation function

A second independent anomalous dimension is hidden in the behavior of the correlation function $G(\vec{r})$

$$\begin{aligned} \text{Dimension: } [G(\vec{r})] &= [\Phi]^2 = [\text{length}]^{2-d}, \\ [G(k)] &= [G(\vec{r})]^2 \cdot [\text{length}]^d = [\text{length}]^d \\ (\text{since } G(\vec{k}) &= \int d^d r G(\vec{r}) e^{i\vec{k} \cdot \vec{r}}) \end{aligned}$$

Consider specifically $T = T_c \Rightarrow \xi \rightarrow \infty$ is not a relevant length scale.
In the absence of another length scale, we must have $G(\vec{k}) \sim k^{-2}$!

Make again a scaling Ansatz assuming that a second microscopic length scale l_0 comes into play: $G(\vec{k}) = k^{-2} f_G(k l_0)$

Consider limit $k \rightarrow 0$ and assume power law: $\lim_{x \rightarrow 0} f_G(x) \sim x^\eta$

$$\text{Then we have } \boxed{G(\vec{k}) \sim k^{-2+\eta}}$$

$\leadsto G$ acquires an additional anomalous dimension η .

Φ acquires an additional anomalous dimension $\eta/2$.

5.4.2 "Mean-field" length scale

Now address next problem:

Why isn't the Josephson relation $(2 - \alpha) = \nu d$ always valid?

(And why does it cross over into the mean-field relation $(2 - \alpha) = \nu d_c$?)

To this end: Inspect dimensions of the parameters r_0, u_0 .

$$\begin{aligned} [r_0] &\sim \text{length}^{-2} \rightarrow \text{length scale } \xi \\ [u_0] &\sim \text{length}^{d-4} \rightarrow \text{new length scale} \end{aligned}$$

Corresponding scaling Ansatz for f_{sing}

$$f_s = \xi^{-d} \hat{f}(\underbrace{u_0 \xi^{4-d}}_{\text{dimensionless}})$$

For $d < 4$: Argument of $\hat{f}(x)$: $x \rightarrow \infty$ for $T \rightarrow T_c$ or $\xi \rightarrow \infty$

For $d > 4$: Argument of $\hat{f}(x)$: $x \rightarrow 0$ for $T \rightarrow T_c$ or $\xi \rightarrow \infty$

$\leadsto$ Suggests solution for our problem:

$$\text{If } \left. \begin{aligned} \lim_{x \rightarrow \infty} \hat{f}(x) &\sim \frac{1}{x} \\ \lim_{x \rightarrow 0} \hat{f}(x) &\sim \text{const} \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} f_s &\sim \xi^{-d} & \text{for } d < 4 \\ f_s &\sim \frac{1}{u_0} \xi^{-4} & \text{for } d > 4 \end{aligned} \right.$$

$\Rightarrow$ This reproduces crossover from $(2 - \alpha) = \nu d$ to $(2 - \alpha) = 4\nu$ at $d = d_c = 4$.

5.4.3 Conclusion from Secs. 5.3 and 5.4

On principle, the dimensional analysis is a good method for constructing scaling hypotheses: It helps to derive relations between "dimensionless" and hence scale invariant quantities. However, this approach is not rigorous, since additional length scales apart from the correlation length exist and may influence the critical exponents. A more rigorous approach is provided by the renormalization group, see next chapter 6.

However, scaling arguments can help to derive theoretical predictions without much effort for a variety of problems. Some examples are given below.

5.5 Other applications of scaling arguments

5.5.1 Finite size scaling

Problem: In simulations (and more and more in experiments), system sizes are not macroscopic, but finite

Example: Simulations with periodic boundary conditions

~> No boundary effects, nevertheless no thermodynamic limit

~> Order parameter distribution $P(m, L)$

Scaling Ansatz:

Dimensions: $[m] = [\xi]^{d_m}$, $[L] = [\xi]$, $[P] = [\xi]^{-d_m}$ (since $\int dm P(m) = 1$)

$$\Rightarrow P(m, L) = \xi^{-d_m} \tilde{P}(m \xi^{-d_m}, L/\xi) = L^{-d_m} \tilde{P}(m L^{-d_m}, L/\xi)$$

(with $\tilde{P}(y, x) = x^{d_m} \tilde{P}(y x^{d_m}, x)$)

Rewrite using $\xi = \hat{\xi} t^{-\nu}$ and $d_m = -\beta/\nu$: $P(m, L) = L^{\beta/\nu} \hat{P}(m L^{\beta/\nu}, t L^{1/\nu})$

Use this to calculate finite size effects on various quantities

Example: Moments of the order parameter

$$\begin{aligned} \langle |m| \rangle_L &= \int dm P(m, L) |m| = L^{-\beta/\nu} f_{\langle |m| \rangle}(t L^{1/\nu}) \\ \langle m^n \rangle_L &= \int dm P(m, L) m^n = L^{-n\beta/\nu} f_{\langle m^n \rangle}(t L^{1/\nu}) \\ &\text{(with } f_{\langle |m| \rangle}(x) = \int dy \hat{P}(y, x) |y|, \quad f_{\langle m^n \rangle}(x) = \int dy \hat{P}(y, x) y^n) \end{aligned}$$

Application: Determining T_c and exponents β, ν from simulations of finite systems. Need simulation data for a set of different sizes L

(a) Binder cumulant method to find T_c

Consider specifically $t = 0$, i.e. $T = T_c$: $\Rightarrow \langle m^{2n} \rangle \sim L^{-2n\beta/\nu} \hat{f}_{\langle m^{2n} \rangle}(0)$

$$\Rightarrow \boxed{U_L = \frac{\langle m^2 \rangle^2}{\langle m^4 \rangle}} \text{ is independent of system size } L!$$

~> Plot U_L vs. T for different L . All curves intersect at $T = T_c$!

(b) Scaling plot method to find β, ν (after finding T_c via (a))

– Guess β, ν and plot $|m| L^{-\beta/\nu}$ versus $t L^{1/\nu}$ for various L

– Vary β and ν until the curves collapse onto one scaling curve (or use a fitting procedure that does this for you).

5.5.2 Scaling hypothesis in polymer physics

System: Linear macromolecules in "good" solvent

Number of repeat units (monomers): N

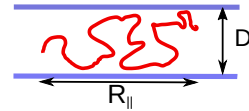
Extension: "Gyration radius": $R_g = aN^\nu$,
with ν : Flory exponent ($\nu = 0.588... \approx 3/5$)

Remark: The calculation of ν is an interesting problem of statistical physics in itself which involves advanced methods of statistical field theory. Polymers in good solvent (so-called self-avoiding walks) can be described by a ϕ^4 theory in the limit where the order parameter ϕ is zero dimensional (de Gennes, 1972))

Assumption (de Gennes): R_g is the only relevant length scale in such systems.
This allows to deduce many relations, which mostly turn out to be correct!

Example: Polymer in a tube of diameter D

Question: What is the lateral extension $R_{\parallel}$?



Scaling Ansatz: $R_{\parallel}(DR_g) = R_g f_{R_{\parallel}}(R_g/D)$

with conditions: $R_{\parallel}(D \rightarrow \infty, R_g) \sim R_g \rightarrow f(x) \xrightarrow{x \rightarrow 0} 1$

$R_{\parallel}(D \rightarrow 0, R_g) \propto N \rightarrow f(x) \xrightarrow{x \rightarrow \infty} x^\mu$

with $R_g^{\mu+1} = N^{\nu(\mu+1)} \stackrel{!}{\sim} N \Rightarrow \mu = \frac{1}{\nu} - 1$

$\leadsto$ In case $D \ll R_g$, one obtains $R_{\parallel} \sim R_g^{\mu+1}/D^\mu$

and hence $\boxed{R_{\parallel} \sim aN(a/D)^{1/\nu-1}}$

(typical 'back of the envelope' scaling calculation!)

(more examples: Exercises)