FIRST EDITION

List provided by John Dambrowski (march 2004)

- On p. 39, $-\ell$. 39: $i+j=p+q-(i-p+j-q) \Longrightarrow i+j=p+q+(i-p)+(j-q)$
- On p. 81 ℓ. 6: $x \mp \mathrm{i}Jx \Longrightarrow x = \mp\mathrm{i}Jx$
- p. 81 ℓ. 11: $g(\mathrm{i}x, \mathrm{i}x) \Longrightarrow g(\mathrm{i}x, \mathrm{i}y)$
- p. 336 ℓ. 7: $\mathrm{i}^w \Longrightarrow -\mathrm{i}^w$
- p. 336 ℓ. -5 and below: $\sigma_p \Longrightarrow \sigma^p$
- p. 337 ℓ. 10 $(\sigma^{p+1})^* : \mathcal{H}^{p+1,w-p-1} \to \mathcal{A}^{0,1}_M(\mathcal{H}^{p,w-p})$ *should read* $(\sigma^p)^* : \mathcal{H}^{p-1,w-p+1} \to \mathcal{A}^{0,1}_M(\mathcal{H}^{p,w-p})$ and on ℓ. 12 one should also replace $(\sigma^{p+1})^*$ by $(\sigma^p)^*$. In accordance with the convention of using subscripts, on pages 340 and 341 all σ_j should have upper scripts

Errors in Chapter 12 (noticed by C. Peters, august 2006)

On p. 327 in the statement and the proof of Cor. 12.2.3 the transpose signs have no meaning and should be deleted: on ℓ -11: $\xi^* = {}^t\bar{\xi} \Longrightarrow \xi^* = \bar{\xi}$, on ℓ -10: $\xi^* = -{}^t\bar{\xi} \Longrightarrow \xi^* = -\bar{\xi}$ and in the proof, ℓ - 3 one has to remove twice the transpose signs s well

Errors have been brought to our attention by Klaus Hertling (september 2006)

- on p. 63 in the two displayed formulas "kernel" and "cokernel" should be switched, i.e. $K=$ kernel [.... $\Longrightarrow$ $K=$ cokernel [..., and likewise Q=cokernel[... $\Longrightarrow$ Q=kernel[....
- on p. 336, in Problem 14.1.1. ii): $\nabla = \nabla^M \Longrightarrow \nabla = \nabla^N$.
- p. 363, ℓ -10 : $C * df = -\cdots \Longrightarrow *df = \ldots$ i.e. the C should be dropped and the sign is $+$. The reason is that C is the identity on functions and $C^{-1} = -C$ on 1-forms.

Errors spotted by S. Müller-Stach and C. Peters (2006)

- On p. 194 the Koszul complex from Prop. 7.1.2 ends with $i_1 : V \otimes S(-1) \to S$ and *not* with $V \otimes S(-1) \to \mathbf{C}$. Indeed the map i_1 surjects onto S^+ with cokernel $S/S^+ \simeq \mathbf{C}$ which shows that it gives a resolution of $\mathbf{C}$ considered as an S-module. Likewise in Problem 7.2.1 (page 199) the last item of the exact sequence should be $\mathcal{O}_W \otimes S^\bullet W$.

- The examples on page 365. The unit ball is $\mathrm{SU}(1,n)/\mathrm{U}(n)$ and *not* $\mathrm{SU}(1,n)/\mathrm{SU}(n)$.

C. Peters reports (2007)

- p. 311, ℓ -6,5. Then $\phi_t(g) = L_g\phi_t(e) = g \cdot \exp(t\xi)$, showing that the one-parameter group acts through *right* multiplication.

- p. 312 In Definition 11.3.1 on should use the *right* V-action. This is consistent with viewing D as a homegeneous space under the *left* G-action; indeed $G \to D$ is a principal V-bundle where V acts from the right.

- p. 314. The formula (11.3.7) should be justified. Indeed, the Maurer-cartan form ω_V is the unique $\mathfrak{v}$-valued 1-form on V which on the left-invariant vector field $(L_v)_*A$, $A \in \mathfrak{v} = T_eV$ equals A. The form $V_p^*\omega$ has this property since $[V_P]_*(L_v)_*A = A^*_{pv}$ on which – by definition – ω at the point pv takes the value A.

  The formula for the adjoint action is $\mathrm{Ad}(v)\xi = v\xi v^{-1}$ and $R_v^*\omega = \mathrm{Ad}(v^{-1})\omega$ which means that the value of ω at $p \cdot v$ evaluated on $(R_v)_*\xi$ equals $v^{-1}\omega_p(\xi)v$.

- p. 315. The statement of Prop. 11.3.10 is confusing. A tangent vector $\xi \in T_mM$ is the tangent at m to some curve γ. A section s of $[W]$ determines a point $s(\gamma(t)) \in [W]_{\gamma(t)}$ which can be represented by a pair $[p_t, w_t]$, $p_t \in P_{\gamma(t)}$, the fiber of P above $\gamma(t)$. The curve γ has a unique parallel lift $\tilde{\gamma}$ through p_t defining parallel transport from the fiber $[W]_{\gamma(t)}$ to a fiber $[W]_{\gamma(t_0)}$ above any other point of the curve γ. For $t_0 = 0$ you get the isomorphism $\theta_t : [W]_{\gamma(t)} \to [W]_m$ and hence, varying t, a curve of vectors $\theta_t[s(\gamma(t)] \in [W]_m$. This curve is called $\vec{s}(t)$ in the statement. In the proof we should take a fixed basis for $[W]_m$ and *not* for W. Alternatively, one has the basis free formula in $[W]_m$:

$$[D_\xi s]\,(m) = \lim_{t\to 0} \frac{\theta_t[s(\gamma(t)] - s(m)}{t}.$$

The parallel frame $\{w_j\}$ obtained from this should be considered as *constant*, i.e. $D_\xi w_j = 0$. The curve $\vec{s}(t)$ can then be identified with a varying column vector in $\mathbb{R}^n$ and the Leibniz rule dictates that $D_\xi(s) \in [W]_m$ under this identification just becomes $\left.\frac{d\vec{s}}{dt}\right|_{t=0} \in \mathbb{R}^n$. The ambiguities here are the choice of the curve γ and the choice of the pair (p_t, w_t) representing $s(\gamma(t))$. However, the curve $\vec{s}(t)$ by construction is not affected by the second type of ambiguity, while differentiating at 0 eliminates the first ambiguity.

- p.316. The discussion leading to the change of frame formula is a bit ambiguous. Let $m \in U$ and $\xi \in T_m M$. Then $\tilde{f}(m) = f(m)v(m) \in P$. Let γ be a curve through m with tangent ξ. Then

$$\begin{aligned}
\tilde{f}_*\xi &= \frac{d}{dt}\left[f(\gamma(t)v(\gamma(t))\right]|_{t=0} \\
&= \frac{d}{dt}\left[f(\gamma(t)v(m)\right]|_{t=0} + \frac{d}{dt}\left[f(m)v(\gamma(t))\right]|_{t=0} \\
&= \left[R_{v(m)}\right]_* (f_*\xi) + \left[V_{\tilde{f}(m)}\right]_* (v_*\xi)
\end{aligned}$$

and hence

$$\begin{aligned}
\tilde{f}^*\omega(\xi) &= \omega(\tilde{f}_*\xi) \\
&= v(m)^{-1}f^*\omega(\xi)v(m) + v^*\omega_V(\xi).
\end{aligned}$$

Next, one has to realize V as a matrix group through the representation $\rho : V \to \mathrm{GL}(W)$. Any choice of basis for W makes it then possible to consider $v : U \to V$ as a matrix-valued function which one keeps writing v. So v is the matrix-function $m \mapsto (x_{ij}v(m))$, where the x_{ij} are the standard coordinates on $\mathrm{End}(W)$ defined by the basis for W.

Since the matrix-valued 1-form $v^*\omega_V$ at m is equal to $[x_{ij}(v(m)]^{-1} \cdot [d(x_{ij}\circ v)]_m$, with this convention we can write $v^*\omega_V = v^{-1}dv$. So, finally,

$$\theta_{\tilde{f}} = v^{-1}\theta_f v + v^{-1}dv.$$

- p. 324. Problem 12.1.1 Read $\mathrm{So}(a+b)$ (this group is already connected), or alternatively, $\mathrm{SO}_0(a,b)$.

- p. 324. Problem 12.1.3 can be solved in two ways. First note that the isotropy group B in $G_\mathbb{C}$ of the Hodge filtration $o \in \check{D} = G_\mathbb{C}/B$ has Lie algebra $\mathfrak{o} \oplus \mathfrak{m}^+$ and hence $\mathfrak{m}^-$ is a complement which gives the complexified tangent space. Alternatively, one can use the characterization of the period domain as an open set in a flag-variety of (partial) flags satisfying the first bilinear relation. The flag-variety consists of a subvariety in the product of Grassmannians of certain subspaces F^i of a fixed complex vector space $H_\mathbb{C}$. At the point F^i the tangent space at the Grassmannian is $T_{F^i} = \mathrm{Hom}(F^i, H_\mathbb{C}/F^i)$. The F^i forming a flag translates into compatibilities between the various $X^i \in \mathrm{Hom}(F^i, H_\mathbb{C}/F^i)$ which in our case are equivalent to saying that the X_i induces maps $Y_{ij} : H^{i,w-i} \to H^{j,w-j}$ with $i < j$. So the (Y_{ij}) can be assembled into a map $Y : H_\mathbb{C} \to H_\mathbb{C}$. The first bilinear relation imposes further restrictions on Y stating that $Y \in \mathfrak{g}_\mathbb{C}$. The upshot is that $Y \in \mathfrak{g}_\mathbb{C}$ corresponds to a holomorphic tangent vector in T_oD if and only if $Y \in \sum_{j<0} \mathfrak{g}^{j,-j}$.

- p. 326, $\ell - 2$ $C \in \mathfrak{g}_\mathbb{C} \implies C \in G_\mathbb{C}$.
- p. 328 In Prop.12.2.5 the metric is in fact invariant under the adjoint action of the full group G.
- p. 328. In Problem 12.2.1 the compact group is the group of complex $2n$ by $2n$ *unitary* matrices X.
- p. 321, middle of the page: "We have canonical identifications $T^{\mathrm{vert}}_{D,o} = \mathfrak{k}$..."
  There is something to explain here. The fibration $\omega : G/V \to G/K$ can be seen concretely as follows. Let $H_\mathbb{C} = \bigoplus_{p+q=w} H^{p,q}$ be the Hodge decomposition corresponding to $o \in D$. Extending scalars, the polarizing form b induces the $(-1)^w$-hermitian form $(x,y) \mapsto h(x,y) = b(x,\bar{y})$ on $H_\mathbb{C}$. Introduce the subspaces $H^+_\mathbb{C} := \bigoplus_{p\equiv 0 \bmod 2} H^{p,q}$ and $H^-_\mathbb{C} := \bigoplus_{p\equiv 1 \bmod 2} H^{p,q}$. In the case of even weight $w = 2v$, the spaces $H^\pm_\mathbb{C}$ are complexifications of real vector spaces $H^\pm$ and the form h is just the polarization on these spaces. The second bilinear relation implies that the form $(-1)^v b$ is positive on H^+ and, viewing G/K as the set of b-isotropic real subspaces of H of dimension $a = \sum_{p\equiv 0 \bmod 2} h^{p,q}$ on which $(-1)^v b$ is positive, and we have

  $$\omega(o) = [H^+] \in G/K.$$

  In the case of odd weight $w = 2v-1$, $\dim H^+_\mathbb{C} = \dim H^-_\mathbb{C}$ and the form h is anti-hermitian. Since the Hodge metric $h_C|H^\pm$ equals $(-1)^v(\pm \mathrm{i})h$, the *hermitian* form $(x,y) \mapsto (-1)^v \mathrm{i} b(x,\bar{y}) = (-1)^v \mathrm{i} h(x,y)$ is positive definite on $H^+_\mathbb{C}$. We may identify G/K as the set of maximal $b_\mathbb{C}$-isotropic subspaces of $H_\mathbb{C}$ on which the latter form is positive and

  $$\omega(o) = [H^+_\mathbb{C}] \in G/K.$$

  In both cases the *complexified* tangent space of G/K at the point $\omega(o)$ is isomorphic to the subspace of $\mathrm{Hom}_\mathbb{C}(H^+_\mathbb{C}, H_\mathbb{C}/H^+_\mathbb{C})$ consisting of $b_\mathbb{C}$-preserving maps $H^+_\mathbb{C} \to H^-_\mathbb{C}$, i.e. $\mathfrak{p}_\mathbb{C} = \bigoplus_{j\equiv 1 \bmod 2} \mathfrak{g}^{-j,j}$. It is a subspace of the complexified tangent space $[T_oD]_\mathbb{C}$ mapping isomorphically to $[T_{[H^+]}(G/K)]_{\mathbb{C}}$ respectively $[T_{[H^+_{\mathbb{C}}]}(G/K)]_{\mathbb{C}}$. Its $b_{\mathbb{C}}$-orthogonal complement $\bigoplus_{j\equiv 0 \bmod 2} \mathfrak{g}^{-j,j} = \mathfrak{k}_{\mathbb{C}}$ is the kernel of the map induced by ω and thus must be the tangent space to the fiber of ω, i.e. $\mathfrak{k}_{\mathbb{C}} = [T_o^{\text{vert}} D]_{\mathbb{C}}$.

- p. 330. The structure equations are completely mangled. They should read

$$\begin{aligned} d\phi + \frac{1}{2}[\phi,\phi] &= -\frac{1}{2}[\sigma,\sigma]^{\mathfrak{v}} \\ d\sigma &= -\frac{1}{2}[\sigma,\sigma]^{\mathfrak{m}}. \end{aligned}$$

- p. 349, $\ell - 4$ to give a V-invariant metric $\Longrightarrow$ to give a G-invariant metric...

- p. 350, $\ell 1$: $\mathfrak{g}_C \Longrightarrow \mathfrak{g}_{\mathbb{C}}$.

- p. 351 in the proof of theorem 13.3.3 compactness of the fiber of $G/V \to G/K$ does not play a role; the Hodge metric on $\mathfrak{m} = T_o^{\text{hol}} D$ transports to a G-invariant metric on $T^{\text{hol}} D = [\mathfrak{m}]$. Since $V \subset K$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a K-invariant splitting, it is certainly V-invariant so that the vector bundle $[\mathfrak{p}] = T^{\text{hor}} D$ is well defined and G-homogeneous. To make curvature calculations on this bundle it suffices therefore to do this at o.

  It should be noted that the curvature calculations are carried out on the frame bundles and so, in order to get the expressions on the tangent bundle one should replace the values in the Lie-algebra by their adjoint actions: the tangent bundle is obtained by the adjoint representation.

- p. 351 $\ell - 3$ A computation ... $\Longrightarrow$ Since $h(\xi,\eta) = \mathrm{Tr}(\xi \circ \eta^*)$ (this is indeed the Hodge metric: on $\mathfrak{p}_{\mathbb{C}}$ the operator θ is multiplication with -1, and complex conjugation is the hermitian conjugate), we find (note the adjoint actions)

$$h([[\xi,\xi^*]\xi],\xi) = \mathrm{Tr}([[\xi,\xi^*],\xi] \circ \xi^*) = \mathrm{Tr}([\xi,\xi^*] \circ [\xi,\xi^*]) = \|[\xi,\xi^*]\|_h.$$

  The last equation comes from the fact that if $\eta = [\xi,\xi^*]$ we have $\eta^* = \eta$. For the holomorphic sectional curvature we thus find

$$K(\xi) = -\frac{1}{2}\|[\xi,\xi^*]\|_h \le 0.$$

- p. 353 In the statement of Alhors' Lemma the condition "Ric $f^*\omega_M \ge f^*\omega_M$" is missing!

SECOND EDITION

Brought to our attention by Andrew Salmon (2018)

In general, asterisks (see p. 76 for example) are rendered in a slightly strange way.

1. p. 7 last paragraph ‘alreaddy’ is misspelled.
2. p. 33 first paragraph ‘proper action action’ is an accidentally repeated word.
3. p. 49 Equation should read $e(\overline{S}) = 2e(\mathbf{P}^2) - e(\overline{B})$ (overline is not over B in text) instead.
4. p. 141 proposition 4.4.1, it should say $\mathrm{Aut}(H_{\mathbf{C}}, b)$ instead.
5. p. 149 switches between putting the bar over the ∂ and over the z.
6. p. 200 forgot a right parentheses on $H^p(S, A^{\cdot,\cdot}(M))$.
7. p. 202 top of the page, unmatched parenthesis on $H^n(\Gamma(N, f_* C^{\cdot}(\mathcal{F})))$. Also, problem 6.4.3 uses f to mean both a map $f : M \to N$ and also a map on sheaves.
8. p. 207 bottom of page skips from $\mathbf{C}e_0$ to $\mathbf{C}e_2$.
9. p. 235 Should say $\Gamma\backslash D$ instead in the equation.
10. p. 416 and 418. There is a missing) in theorem 15.2.9 and at the bottom of the page 418
11. p. 429 There is a repeated word ‘domain.’
12. p. 442 G_Z should have a boldface $\mathbf{Z}$.

C. Peters (2018)

About Lemma 13.3.1.

Note that In the statement the Higgs bundle $\mathcal{H}$ is the Deligne extension of a logarithmic complex VHS over (S, Σ) and the holomorphic section s is a *global* section of this Deligne extension, i.e. a section over the compact curve S. In the proof the last few lines are misstated:
"Conversely, a holomorphic section ..." $\longrightarrow$ "Conversely, a flat section induces a holomorphic section on $\mathcal{H}_{\mathrm{Hdg}}$ and then we see that $\sigma(s) = 0$ by type considerations.